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THE FOUNDATIONS OF PHYSICS (FIRST COMMUNICATION)

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The vast problems posed by Einstein[1] as well as his ingeniously conceived methods of solution, and the far-reaching ideas and formation of novel concepts by means of which Mie[2] constructs his electrodynamics, have opened new paths for the investigation into the foundations of physics.

In the following—in the sense of the axiomatic method—I would like to develop, essentially from two simple axioms, a new system of basic equations of physics, of ideal beauty and containing, I believe, *simultaneously* the solution to the problems of Einstein and of Mie. I reserve for later communications the detailed development and particularly the special application of my basic equations to the fundamental questions of the theory of electricity.

Let w_s $(s = 1, 2, 3, 4)$ be any coordinates labeling the world's points essentially uniquely—the so-called world parameters (most general spacetime coordinates). The quantities characterizing the events at w_s shall be:

1. The ten gravitational potentials $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ first introduced by Einstein, having the character of a symmetric tensor with respect to an arbitrary transformation of the world parameters w_s;
2. The four electrodynamic potentials q_s having the character of a vector in the same sense. |

Physical processes do not proceed in an arbitrary way, rather they are governed by [396]
the following two axioms:

1 *Sitzungsber. d. Berliner Akad.* 1914, 1030; 1915, 778, 799, 831, 844.
2 *Ann. d. Phys.* 1912, Vol. 37, 511; Vol. 39, 1; 1913, vol. 40, 1.

Jürgen Renn (ed.). *The Genesis of General Relativity,* Vol. 4
Gravitation in the Twilight of Classical Physics: The Promise of Mathematics.

Axiom I (Mie's axiom of the world function[3]): *The law governing physical processes is determined through a world function* H, *that contains the following arguments:*

$$g_{\mu\nu}, \qquad g_{\mu\nu l} = \frac{\partial g_{\mu\nu}}{\partial w_l}, \qquad g_{\mu\nu lk} = \frac{\partial^2 g_{\mu\nu}}{\partial w_l \partial w_k}, \tag{1}$$

$$q_s, \qquad q_{sl} = \frac{\partial q_s}{\partial w_l} \qquad (l, k = 1, 2, 3, 4), \tag{2}$$

where the variation of the integral

$$\int H \sqrt{g} d\omega$$

$$(g = |g_{\mu\nu}|, \quad d\omega = dw_1 dw_2 dw_3 dw_4)$$

must vanish for each of the fourteen potentials $g_{\mu\nu}, q_s$.

Clearly the arguments (1) can be replaced by the arguments [1]

$$g^{\mu\nu}, \qquad g_l^{\mu\nu} = \frac{\partial g^{\mu\nu}}{\partial w_l}, \qquad g_{lk}^{\mu\nu} = \frac{\partial^2 g^{\mu\nu}}{\partial w_l \partial w_k}, \tag{3}$$

where $g^{\mu\nu}$ is the subdeterminant of the determinant g with respect to its element $g_{\mu\nu}$, divided by g.

Axiom II (axiom of general invariance[4]): *The world function* H *is invariant with respect to an arbitrary transformation of the world parameters* w_s.

Axiom II is the simplest mathematical expression of the demand that the interlinking of the potentials $g_{\mu\nu}, q_s$ is by itself entirely independent of the way one chooses to label the world's points by means of world parameters.

The guiding motive for constructing my theory is provided by the following theorem, the proof of which I shall present elsewhere. |

[397] Theorem I. If J is an invariant under arbitrary transformation of the four world parameters, containing n quantities and their derivatives, and if one forms from

$$\delta \int J g d\omega = 0$$

the n variational equations of Lagrange with respect to those n quantities, then in this invariant system of n differential equations for the n quantities there are always

3 Mie's world functions do not contain exactly these arguments; in particular the usage of the arguments (2) goes back to Born. However, what is characteristic of Mie's electrodynamics is precisely the introduction and use of such a world function in Hamilton's principle.

4 Orthogonal invariance was already postulated by Mie. In the axiom II formulated above, Einstein's fundamental basic idea of general invariance finds its simplest expression, even if Hamilton's principle plays only a subsidiary role with Einstein, and his functions H are by no means general invariants, and also do not contain the electric potentials.

four that are a consequence of the remaining $n-4$ —in this sense, that among the n differential equations and their total derivatives there are always four linear and mutually independent combinations that are satisfied identically.

Concerning the differential quotients with respect to $g^{\mu\nu}$, $g_k^{\mu\nu}$, $g_{kl}^{\mu\nu}$ occurring in (4) and subsequent formulas, let us note once for all that, due to the symmetry in μ, ν on the one hand and in k, l on the other, the differential quotients with respect to $g^{\mu\nu}$, $g_k^{\mu\nu}$ are to be multiplied by 1 resp. $\frac{1}{2}$, according as $\mu = \nu$ resp. $\mu \neq \nu$, further the differential quotients with respect to $g_{kl}^{\mu\nu}$ are to be multiplied by 1 resp. $\frac{1}{2}$ resp. $\frac{1}{4}$, according as $\mu = \nu$ and $k = l$ resp. $\mu = \nu$ and $k \neq l$ or $\mu \neq \nu$ and $k = l$ resp. $\mu \neq \nu$ and $k \neq l$.

Axiom I implies first for the ten gravitational potentials $g^{\mu\nu}$ the ten Lagrangian differential equations

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} = 0, \qquad (\mu, \nu = 1, 2, 3, 4) \qquad (4)$$

and secondly for the four electrodynamic potentials q_s the four Lagrangian differential equations

$$\frac{\partial\sqrt{g}H}{\partial q_h} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial q_{hk}} = 0, \qquad (h = 1, 2, 3, 4). \qquad (5)$$

We denote the left sides of the equations (4), (5) respectively by

$$[\sqrt{g}H]_{\mu\nu}, \qquad [\sqrt{g}H]_h$$

for short.

Let us call equations (4) the fundamental equations of gravitation, and equations (5) the fundamental electrodynamic equations, or generalized Maxwell equations. Due to the theorem stated above, the four equations (5) can be viewed as a consequence of equations (4), that is, because of that mathematical theorem we can directly make the claim *that in the sense as explained the electrodynamic phenomena are effects of gravitation*. I regard this | insight as the simple and very surprising solution of the problem of Riemann, who was the first to search for a theoretical connection between gravitation and light. [398]

In the following we use the easily proved fact that, if p^j $(j = 1, 2, 3, 4)$ is an *arbitrary* contravariant vector, the expression

$$p^{\mu\nu} = \sum_s (g_s^{\mu\nu} p^s - g^{\mu s} p_s^\nu - g^{\nu s} p_s^\mu), \qquad \left(p_s^j = \frac{\partial p^j}{\partial w_s}\right)$$

represents a symmetric contravariant tensor, and the expression

$$p_l = \sum_s (q_{ls} p^s + q_s p_l^s)$$

represents a covariant vector.

To proceed we establish two mathematical theorems, which express the following:

Theorem II. If J is an invariant depending on $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{kl}^{\mu\nu}$, q_s, q_{sk}, then the following is always identically true in all arguments and for every arbitrary contravariant vector p^s: [2]

$$\sum_{\mu,\nu,l,k}\left(\frac{\partial J}{\partial g^{\mu\nu}}\Delta g^{\mu\nu}+\frac{\partial J}{\partial g_l^{\mu\nu}}\Delta g_l^{\mu\nu}+\frac{\partial J}{\partial g_{kl}^{\mu\nu}}\Delta g_{kl}^{\mu\nu}\right)$$
$$+\sum_{s,k}\left(\frac{\partial J}{\partial q_s}\Delta q_s+\frac{\partial J}{\partial q_{sk}}\Delta q_{sk}\right) = 0,$$

where

$$\Delta g^{\mu\nu} = \sum_m (g^{\mu m}p_m^\nu + g^{\nu m}p_m^\mu),$$
$$\Delta g_l^{\mu\nu} = -\sum_m g_m^{\mu\nu}p_l^m + \frac{\partial \Delta g^{\mu\nu}}{\partial w_l},$$
$$\Delta g_{lk}^{\mu\nu} = -\sum_m (g_m^{\mu\nu}p_{lk}^m + g_{lm}^{\mu\nu}p_k^m + g_{km}^{\mu\nu}p_l^m) + \frac{\partial^2 \Delta g^{\mu\nu}}{\partial w_l \partial w_k},$$
$$\Delta q_s = -\sum_m q_m p_s^m,$$
$$\Delta q_{sk} = -\sum_m q_{sm}p_k^m + \frac{\partial \Delta q_s}{\partial w_k}.$$

This theorem II can also be formulated as follows:

If J is an invariant and p^s and arbitrary vector as above, then the identity holds

$$\sum_s \frac{\partial J}{\partial w_s}p^s = PJ, \tag{6}$$

[399] | where we have put

$$P = P_g + P_q,$$

with

$$P_g = \sum_{\mu,\nu,l,k}\left(p^{\mu\nu}\frac{\partial}{\partial g^{\mu\nu}} + p_l^{\mu\nu}\frac{\partial}{\partial g_l^{\mu\nu}} + p_{lk}^{\mu\nu}\frac{\partial}{\partial g_{lk}^{\mu\nu}}\right)$$
$$P_q = \sum_{l,k}\left(p_l\frac{\partial}{\partial q_l} + p_{lk}\frac{\partial}{\partial q_{lk}}\right),$$

and used the abbreviations:

$$p_k^{\mu\nu} = \frac{\partial p^{\mu\nu}}{\partial w_k}, \qquad p_{kl}^{\mu\nu} = \frac{\partial^2 p^{\mu\nu}}{\partial w_k \partial w_l}, \qquad p_{lk} = \frac{\partial p_l}{\partial w_k}.$$

The proof of (6) follows easily; for this identity is obviously correct if p^s is a constant vector, and from this it follows in general because of its invariance.

Theorem III. If J is an invariant depending *only* on $g^{\mu\nu}$ and their derivatives, and if, as above, the variational derivatives of $\sqrt{g}J$ with respect to $g^{\mu\nu}$ are denoted by $[\sqrt{g}J]_{\mu\nu}$ then the expression—where $h^{\mu\nu}$ is understood to be any contravariant tensor—

$$\frac{1}{\sqrt{g}} \sum_{\mu,\nu} [\sqrt{g}J]_{\mu\nu} h^{\mu\nu}$$

represents an invariant; if we substitute in this sum in place of $h^{\mu\nu}$ the particular tensor $p^{\mu\nu}$ and write

$$\sum_{\mu,\nu} [\sqrt{g}J]_{\mu\nu} p^{\mu\nu} = \sum_{s,l} (i_s p^s + i_s^l p_l^s),$$

where then the expressions

$$i_s = \sum_{\mu,\nu} [\sqrt{g}J]_{\mu\nu} g_s^{\mu\nu},$$

$$i_s^l = -2\sum_{\mu} [\sqrt{g}J]_{\mu s} g^{\mu l}$$

depend only on the $g^{\mu\nu}$ and their derivatives, then we have

$$i_s = \sum_l \frac{\partial i_s^l}{\partial w_l} \tag{7}$$

in the sense that this equation is satisfied identically for all arguments, that is for the $g^{\mu\nu}$ and their derivatives.

For the proof we consider the integral

$$\int J\sqrt{g}\,d\omega, \qquad d\omega = dw_1 dw_2 dw_3 dw_4$$

to be taken over a finite piece of the four dimensional world. | Further, let p^s be a vector that vanishes together with its derivatives on the three dimensional surface of that piece of the world. Due to $P = P_g$ the last formula of the next page implies [400]

$$P_g(\sqrt{g}J) = \sum_s \frac{\partial \sqrt{g} J p^s}{\partial w_s};$$

this results in

$$\int P_g(J\sqrt{g})d\omega = 0$$

and due to the way the Lagrangian derivative is formed we accordingly also have

$$\int \sum_{\mu,\nu} [\sqrt{g}J]_{\mu\nu} p^{\mu\nu} d\omega = 0.$$

Introduction of i_s, i_s^l into this identity finally shows that

$$\int \left(\sum_l \frac{\partial i_s^l}{\partial w_l} - i_s \right) p^s d\omega = 0$$

and therefore also that the assertion of our theorem is correct.

The most important aim is now the formulation of the concept of energy, and the derivation of the energy theorem solely on the basis of the two axioms I and II.

For this purpose we first form:

$$P_g(\sqrt{g}H) = \sum_{\mu,\nu,k,l} \left(\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} p^{\mu\nu} + \frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} p_k^{\mu\nu} + \frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} p_{kl}^{\mu\nu} \right).$$

Now $\dfrac{\partial H}{\partial g_{kl}^{\mu\nu}}$ is a mixed tensor of fourth rank, so if one puts

$$A_k^{\mu\nu} = p_k^{\mu\nu} + \sum_\rho \left(\begin{Bmatrix} k\rho \\ \mu \end{Bmatrix} p^{\rho\nu} + \begin{Bmatrix} k\rho \\ \nu \end{Bmatrix} p^{\rho\mu} \right),$$

$$\begin{Bmatrix} k\rho \\ \mu \end{Bmatrix} = \frac{1}{2} \sum_\sigma g^{\mu\sigma}(g_{k\sigma\rho} + g_{\rho\sigma k} - g_{k\rho\sigma}),$$

the expression

$$a^l = \sum_{\mu,\nu,k} \frac{\partial H}{\partial g_{kl}^{\mu\nu}} A_k^{\mu\nu} \qquad (8)$$

becomes a contragredient vector.

Hence if we form the expression

$$P_g(\sqrt{g}H) - \sum_l \frac{\partial\sqrt{g}a^l}{\partial w_l}$$

[401] then this no longer contains the second derivatives $p_{kl}^{\mu\nu}$ and | therefore has the form

$$\sqrt{g} \sum_{\mu,\nu,k} (B_{\mu\nu} p^{\mu\nu} + B_{\mu\nu}^k p_k^{\mu\nu}),$$

where

$$B^k_{\mu\nu} = \sum_{\rho, l}\left(\frac{\partial H}{\partial g^{\mu\nu}_k} - \frac{\partial}{\partial w_l}\frac{\partial H}{\partial g^{\mu\nu}_{kl}} - \frac{\partial H}{\partial g^{\rho\nu}_{kl}}\begin{Bmatrix} l\mu \\ \rho \end{Bmatrix} - \frac{\partial H}{\partial g^{\mu\rho}_{kl}}\begin{Bmatrix} l\nu \\ \rho \end{Bmatrix}\right)$$

is again a mixed tensor.

Now we form the vector

$$b^l = \sum_{\mu, \nu} B^l_{\mu\nu} p^{\mu\nu}, \tag{9}$$

and obtain from it

$$P_g(\sqrt{g}H) - \sum_l \frac{\partial\sqrt{g}(a^l + b^l)}{\partial w_l} = \sum_{\mu, \nu}[\sqrt{g}H]_{\mu\nu} p^{\mu\nu}. \tag{10}$$

On the other hand we form

$$P_q(\sqrt{g}H) = \sum_{k, l}\left(\frac{\partial\sqrt{g}H}{\partial q_k} p_k + \frac{\partial\sqrt{g}H}{\partial q_{kl}} p_{kl}\right);$$

then $\frac{\partial H}{\partial q_{kl}}$ is a tensor and the expression

$$c^l = \sum_k \frac{\partial H}{\partial q_{kl}} p_k \tag{11}$$

therefore represents a contragredient vector. Correspondingly, as above, we obtain

$$P_q(\sqrt{g}H) - \sum_l \frac{\partial\sqrt{g}c^l}{dw_l} = \sum_k [\sqrt{g}H]_k p_k. \tag{12}$$

Now we note the basic equations (4) and (5), and conclude by adding (10) and (12):

$$P(\sqrt{g}H) = \sum_l \frac{\partial\sqrt{g}(a^l + b^l + c^l)}{dw_l}.$$

But we have

$$P(\sqrt{g}H) = \sqrt{g}PH + H\sum_{\mu, \nu}\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} p^{\mu\nu}$$

$$= \sqrt{g}PH + H\sum_s\left(\frac{\partial\sqrt{g}}{\partial w_s} p^s + \sqrt{g}p^s_s\right),$$

and thus, due to identity (6)

$$P(\sqrt{g}H) = \sqrt{g}\sum_s \frac{\partial H}{\partial w_s}p^s + H\sum_s\left(\frac{\partial\sqrt{g}}{\partial w_s}p^s + \sqrt{g}p^s_s\right) = \sum_s \frac{\partial\sqrt{g}Hp^s}{\partial w_s}.$$

[402] | From this we finally obtain the invariant equation

$$\sum_l \frac{\partial}{\partial w_l}\sqrt{g}(Hp^l - a^l - b^l - c^l) = 0.$$

Now we note that

$$\frac{\partial H}{\partial q_{lk}} - \frac{\partial H}{\partial q_{kl}}$$

is a skew symmetric contravariant tensor; consequently

$$d^l = \frac{1}{2\sqrt{g}}\sum_{k,\,s}\frac{\partial}{\partial w_k}\left\{\left(\frac{\partial\sqrt{g}H}{\partial q_{lk}} - \frac{\partial\sqrt{g}H}{\partial q_{kl}}\right)p^s q_s\right\} \tag{13}$$

becomes a contravariant vector, which evidently satisfies the identity

$$\sum_l \frac{\partial\sqrt{g}d^l}{\partial w_l} = 0.$$

Let us now define

$$e^l = Hp^l - a^l - b^l - c^l - d^l \tag{14}$$

as the *energy vector, then the energy vector is a contravariant vector, which moreover depends linearly on the arbitrarily chosen vector* p^s, *and satisfies identically for that choice of this vector* p^s *the invariant energy equation*

$$\sum_l \frac{\partial\sqrt{g}e^l}{\partial w_l} = 0.$$

As far as the world function H is concerned, further axioms are needed to determine its choice in a unique way. If the gravitational field equations are to contain only second derivatives of the potentials $g^{\mu\nu}$, then H must have the form

$$H = K + L$$

where K is the invariant that derives from the Riemannian tensor (curvature of the four-dimensional manifold)

$$K = \sum_{\mu,\,\nu} g^{\mu\nu}K_{\mu\nu}$$

$$K_{\mu\nu} = \sum_{\kappa}\left(\frac{\partial}{\partial w_{\nu}}\begin{Bmatrix}\mu\kappa\\ \kappa\end{Bmatrix} - \frac{\partial}{\partial w_{\kappa}}\begin{Bmatrix}\mu\nu\\ \kappa\end{Bmatrix}\right) + \sum_{\kappa,\lambda}\left(\begin{Bmatrix}\mu\kappa\\ \lambda\end{Bmatrix}\begin{Bmatrix}\lambda\nu\\ \kappa\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \lambda\end{Bmatrix}\begin{Bmatrix}\lambda\kappa\\ \kappa\end{Bmatrix}\right)$$

and where L depends only on $g^{\mu\nu}$, $g_l^{\mu\nu}$, q_s, q_{sk}. Finally we make the simplifying assumption in the following, that L does not contain the $g_l^{\mu\nu}$. |

Next we apply theorem II to the invariant L and obtain [403]

$$\sum_{\mu,\nu,m}\frac{\partial L}{\partial g^{\mu\nu}}(g^{\mu m}p_m^{\nu} + g^{\nu m}p_m^{\mu}) - \sum_{s,m}\frac{\partial L}{\partial q_s}q_m p_s^m - \sum_{s,k,m}\frac{\partial L}{\partial q_{sk}}(q_{sm}p_k^m + q_{mk}p_s^m + q_m p_{sk}^m) = 0. \tag{15}$$

Equating to zero the coefficient of p_{sk}^m on the left produces the equation

$$\left(\frac{\partial L}{\partial q_{sk}} + \frac{\partial L}{\partial q_{ks}}\right)q_m = 0$$

or

$$\frac{\partial L}{\partial q_{sk}} + \frac{\partial L}{\partial q_{ks}} = 0, \tag{16}$$

that is, the derivatives of the electrodynamic potentials q_s occur only in the combinations

$$M_{ks} = q_{sk} - q_{ks}.$$

Thus we learn that under our assumptions the invariant L depends, besides on the potentials $g_{\mu\nu}$, q_s, only on the components of the skew symmetric invariant tensor

$$M = (M_{ks}) = \mathrm{Curl}(q_s),$$

that is, of the so-called electromagnetic six vector. *This result, which determines the character of Maxwell's equations in the first place, here derives essentially as a consequence of the general invariance, that is, on the basis of axiom II.*

If we put the coefficient of p_m^{ν} on the left of identity (15) equal to zero, we obtain, using (16)

$$2\sum_{\mu}\frac{\partial L}{\partial g^{\mu\nu}}g^{\mu m} - \frac{\partial L}{\partial q_m}q_{\nu} - \sum_{s}\frac{\partial L}{\partial M_{ms}}M_{\nu s} = 0, \qquad (\mu = 1, 2, 3, 4). \tag{17}$$

This equation admits an important transformation of the electromagnetic energy, that is the part of the energy vector that comes from L. Namely, this part results from (11), (13), (14) as follows:

$$Lp^l - \sum_k \frac{\partial L}{\partial q_{kl}} p_k - \frac{1}{2\sqrt{g}} \sum_{k,s} \frac{\partial}{\partial w_k} \left\{ \left(\frac{\partial \sqrt{g} L}{\partial q_{lk}} - \frac{\partial \sqrt{g} L}{\partial q_{kl}} \right) p^s q_s \right\}.$$

Because of (16) and by noting (5) this expression becomes |

[404]

$$\sum_{s,k} \left(L\delta_s^l - \frac{\partial L}{\partial M_{lk}} M_{sk} - \frac{\partial L}{\partial q_l} q_s \right) p^s \tag{18}$$

$$(\delta_s^l = 0, \quad l \neq s; \quad \delta_s^s = 1)$$

so because of (17) it equals

$$-\frac{2}{\sqrt{g}} \sum_{\mu, s} \frac{\partial \sqrt{g} L}{\partial g^{\mu s}} g^{\mu l} p^s. \tag{19}$$

Because of the formulas (21) to be developed below we see from this in particular that the electromagnetic energy, and therefore also the total energy vector e^l can be expressed through K alone, so that only the $g^{\mu\nu}$ and their derivatives, but not the q_s and their derivatives occur in it. If one takes the limit

$$g_{\mu\nu} = 0, \qquad (\mu \neq \nu)$$

$$g_{\mu\mu} = 1$$

in expression (18), then this limit agrees exactly with what Mie has proposed in his electrodynamics: *Mie's electromagnetic energy tensor is nothing but the generally invariant tensor that results from differentiation of the invariant L with respect to the gravitational potentials $g^{\mu\nu}$ in that limit*—a circumstance that gave me the first hint of the necessary close connection between Einstein's general relativity theory and Mie's electrodynamics, and which convinced me of the correctness of the theory here developed.

It remains to show directly how with the assumption

$$H = K + L \tag{20}$$

the generalized Maxwell equations (5) put forth above are entailed by the gravitational equations (4).

Using the notation introduced earlier for the variational derivatives with respect to the $g^{\mu\nu}$, the gravitational equations, because of (20), take the form

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial \sqrt{g} L}{\partial g^{\mu\nu}} = 0. \tag{21}$$

The first term on the left hand side becomes

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu}\right),$$

| as follows easily without calculation from the fact that $K_{\mu\nu}$, apart from $g_{\mu\nu}$, is the only tensor of second rank and K the only invariant, that can be formed using only the $g^{\mu\nu}$ and their first and second differential quotients, $g_k^{\mu\nu}$, $g_{kl}^{\mu\nu}$. [405]

The resulting differential equations of gravitation appear to me to be in agreement with the grand concept of the theory of general relativity established by Einstein in his later treatises.[5]

Further, if we denote in general the variational derivatives of $\sqrt{g}J$ with respect to the electrodynamic potential q_h as above by

$$[\sqrt{g}J]_h = \frac{\partial\sqrt{g}J}{\partial q_h} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}J}{\partial q_{hk}},$$

then the basic electromagnetic equations assume the form, due to (20)

$$[\sqrt{g}L]_h = 0. \tag{22}$$

Since K is an invariant that depends only on the $g^{\mu\nu}$ and their derivatives, by theorem III the equation (7) holds identically, with

$$i_s = \sum_{\mu,\nu}[\sqrt{g}K]_{\mu\nu}g_s^{\mu\nu} \tag{23}$$

and

$$i_s^l = -2\sum_{\mu}[\sqrt{g}K]_{\mu s}g^{\mu l}, \qquad (\mu = 1, 2, 3, 4). \tag{24}$$

Due to (21) and (24), (19) equals $-\frac{1}{\sqrt{g}}i_\nu^m$. By differentiating with respect to w_m and summing over m we obtain because of (7)

$$\begin{aligned} i_\nu &= \sum_m \frac{\partial}{\partial w_m}\left(-\sqrt{g}L\delta_\nu^m + \frac{\partial\sqrt{g}L}{\partial q_m}q_\nu + \sum_s \frac{\partial\sqrt{g}L}{\partial M_{sm}}M_{s\nu}\right) \\ &= -\frac{\partial\sqrt{g}L}{\partial w_\nu} + \sum_m \left\{ q_\nu \frac{\partial}{\partial w_m}\left([\sqrt{g}L]_m + \sum_s \frac{\partial}{\partial w_s}\frac{\partial\sqrt{g}L}{\partial q_{ms}}\right) \right. \\ &\quad \left. + q_{\nu m}\left([\sqrt{g}L]_m + \sum_s \frac{\partial}{\partial w_s}\frac{\partial\sqrt{g}L}{\partial q_{ms}}\right) \right\} \\ &\quad + \sum_s \left([\sqrt{g}L]_s - \frac{\partial\sqrt{g}L}{\partial q_s}\right)M_{s\nu} + \sum_{s,m}\frac{\partial\sqrt{g}L}{\partial M_{sm}}\frac{\partial M_{s\nu}}{\partial w_m}, \end{aligned}$$

5 Loc. cit. *Berliner Sitzungsber.* 1915.

since of course

$$\frac{\partial\sqrt{g}L}{\partial q_m} = [\sqrt{g}L]_m + \sum_s \frac{\partial}{\partial w_s}\frac{\partial\sqrt{g}L}{\partial q_{ms}}$$

[406] | and[3]

$$-\sum_m \frac{\partial}{\partial w_m}\frac{\partial\sqrt{g}L}{\partial q_{sm}} = [\sqrt{g}L]_s - \frac{\partial\sqrt{g}L}{\partial q_s}.$$

Now we take into account that because of (16) we have

$$\sum_{m,s}\frac{\partial^2}{\partial w_m \partial w_s}\frac{\partial\sqrt{g}L}{\partial q_{ms}} = 0,$$

and then obtain by suitably collecting terms

$$\begin{aligned} i_\nu = &-\frac{\partial\sqrt{g}L}{\partial w_\nu} + \sum_m\left(q_\nu\frac{\partial}{\partial w_m}[\sqrt{g}L]_m + M_{m\nu}[\sqrt{g}L]_m\right) \\ &+ \sum_m \frac{\partial\sqrt{g}L}{\partial q_m}q_{m\nu} + \sum_{s,m}\frac{\partial\sqrt{g}L}{\partial M_{sm}}\frac{\partial M_{s\nu}}{\partial w_m}. \end{aligned} \tag{25}$$

On the other hand we have

$$-\frac{\partial\sqrt{g}L}{\partial w_\nu} = -\sum_{s,m}\frac{\partial\sqrt{g}L}{\partial g^{sm}}g_\nu^{sm} - \sum_m\frac{\partial\sqrt{g}L}{\partial q_m}q_{m\nu} - \sum_{m,s}\frac{\partial\sqrt{g}L}{\partial q_{ms}}\frac{\partial q_{ms}}{\partial w_\nu}.$$

The first term on the right is nothing other than i_ν because of (21) and (23). The last term on the right proves to be equal and opposite to the last term on the right of (25); namely, we have

$$\sum_{s,m}\frac{\partial\sqrt{g}L}{\partial M_{sm}}\left(\frac{\partial M_{s\nu}}{\partial w_m} - \frac{\partial q_{ms}}{\partial w_\nu}\right) = 0, \tag{26}$$

since the expression

$$\frac{\partial M_{s\nu}}{\partial w_m} - \frac{\partial q_{ms}}{\partial w_\nu} = \frac{\partial^2 q_\nu}{\partial w_s \partial w_m} - \frac{\partial^2 q_s}{\partial w_\nu \partial w_m} - \frac{\partial^2 q_m}{\partial w_\nu \partial w_s}$$

is symmetric in s, m, and the first factor under the summation sign in (26) turns out to be skew symmetric in s, m.

Consequently (25) entails the equation

$$\sum_m\left(M_{m\nu}[\sqrt{g}L]_m + q_\nu\frac{\partial}{\partial w_m}[\sqrt{g}L]_m\right) = 0; \tag{27}$$

that is, from the gravitational equations (4) there follow indeed the four mutually independent linear combinations (27) of the basic electrodynamic equations (5) and their first derivatives. *This is the exact mathematical expression of the statement claimed in general above concerning the character of electrodynamics as a consequence of gravitation.* |

According to our assumption L should not depend on the derivatives of the $g^{\mu\nu}$; [407]
therefore L must be a function of certain four general invariants, which correspond to the special orthogonal invariants given by Mie, and of which the two simplest ones are these:

$$Q = \sum_{k, l, m, n} M_{mn} M_{lk} g^{mk} g^{nl}$$

and

$$q = \sum_{k, l} q_k q_l g^{kl}.$$

The simplest and most straightforward ansatz for L, considering the structure of K, is also that which corresponds to Mie's electrodynamics, namely

$$L = \alpha Q + f(q)$$

or, following Mie even more closely:

$$L = \alpha Q + \beta q^3,$$

where $f(q)$ denotes any function of q, and α, β are constants.

As one can see, the few simple assumptions expressed in axioms I and II suffice with appropriate interpretation to establish the theory: through it not only are our views of space, time, and motion fundamentally reshaped in the sense explained by Einstein, but I am also convinced that through the basic equations established here the most intimate, presently hidden processes in the interior of the atom will receive an explanation, and in particular that generally a reduction of all physical constants to mathematical constants must be possible—even as in the overall view thereby the possibility approaches that physics in principle becomes a science of the type of geometry: surely the highest glory of the axiomatic method, which as we have seen takes the powerful instruments of analysis, namely variational calculus and theory of invariants, into its service.

EDITORIAL NOTES

[1] The index $_l$ of $\partial\omega_l$ in the denominator of the third equation is missing in the original text.

[2] The subscript $_{sk}$ in the denominator of ∂q_{sk} is missing in the original text.

[3] The subscript $_s$ in the term $[\sqrt{g}L]_s$ is missing in the original text.