DAVID HILBERT AND THE ORIGIN OF THE "SCHWARZSCHILD SOLUTION"

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ABSTRACT. The very early dismissal of Schwarzschild's original solution and manifold, and the rise, under Schwarzschild's name, of the inequivalent solution and manifold found instead by Hilbert, are scrutinised and commented upon, in the light of the subsequent occurrences.

It is reminded that Hilbert's manifold suffers from two defects, that are absent in Schwarzschild's manifold. It does not admit a consistent drawing of the arrow of time, and it allows for an invariant, local, intrinsic singularity in its interior. The former defect is remedied by the change of topology of the extensions proposed by Synge, Kruskal and Szekeres. The latter persists unaffected in the extensions, since it is of local character.

1. INTRODUCTION

There is, indisputably, no issue in Einstein's theory of general relativity that has been so accurately scrutinized, and by so many relativists in so many decades, like the "Schwarzschild solution". Innumerable research articles have been devoted to its study, and still are at the present day. Any textbook of relativity, either introductory or advanced, cannot help dedicating one chapter or more to the derivation of this paradigmatic solution of Einstein's field equations for the vacuum, followed by a discussion of the obtained result and of the theoretical predictions stemming from it. In the books published after 1970 (with some notable exceptions) one more chapter is then devoted to the task of removing, through the Kruskal maximal extension, the singularity that the metric components in the famous "Schwarzschild" expression for the interval

$$(1) \qquad \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2),$$

where

$$(2) \qquad\qquad\qquad 0 < r < \infty,$$

exhibit, in "Schwarzschild" coordinates, at $r = 2m$. The reader is always ensured that this is a spurious singularity, devoid of local physical meaning, due only to the bad choice of the coordinate system done by Schwarzschild.

It is therefore a bit surprising to learn, through the direct reading of the original "Massenpunkt" paper [1], that Karl Schwarzschild never wrote a solution given by equations (1) and (2), nor a solution whose manifold was

in one to one correspondence with the latter. Even worse: it turns out that, due to his method of solution, he had the possibility to write a manifold in one to one correspondence with the manifold described by (1) and (2), but deliberately refused to do so.

In fact, after the Minkowskian boundary conditions at the spatial infinity have been satisfied, Schwarzschild's original solution appears to contain still two constants of integration, instead of the single one that appears in (1) and (2). One of the constants has to do with the active gravitational mass, and Schwarzschild chose it by appealing to Newton; the second one determines the position of the inner border of the manifold. Schwarzschild therefore needed an additional postulate in order to fix this second constant. By appealing to the requirement of continuity for the components of the metric in the range between the inner border and the spatial infinity, Schwarzschild chose his second constant in such a way as to position the singularity that brings his name just on the inner border of the manifold.

This singular outcome of the perusal of Schwarzschild's original paper [1] will not be expounded here any further, because it has already been scrutinized in the Note [2] that accompanies a recent English translation [3] of the "Massenpunkt" paper. One has rather answering here the ensuing questions: how did it happen that the manifold described by (1) *cum* (2) was called "Schwarzschild solution", and why and when the original solution with two constants of integration, hence with the need for an additional postulate, was forgotten[1]?

2. Frank's review of the "Massenpunkt" paper

It is remarkable that the substitution of the "Schwarzschild" solution (1,2) for the original one [1] war a very early occurrence, certainly easied by the premature death of Karl Schwarzschild. The seeds of oblivion were already cast in the review with which Philipp Frank presented [7] Schwarzschild's "Massenpunkt" paper to the community of the mathematicians. An English translation of the review is reported in Appendix A . The interested reader is invited to compare that necessarily concise account with the original paper [1]. In this way one can appreciate that Frank's review faithfully extracts several relevant points of Schwarzschild's achievement by accurately following the letter of the text. For the sake of conciseness, however, two facts were completely left in the shadow. Their omission might have appeared marginal to Frank at the time the review was written, but it became crucial soon afterwards, when the rederivations of the Schwarzschild solution by Droste, Hilbert and Weyl appeared [8, 9, 10] in print. And today, if one reads Frank's account without having previously perused Schwarzschild's paper [1], one by no means understands the rationale of Schwarzschild's

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[1]The memory of Schwarzschild's original solution was rekindled at the end of the last century by the works [4, 5, 6] of L. S. Abrams and C. L. Pekeris.

procedure, and why the manifold found by him happens to be inequivalent to the one found in particular by David Hilbert.

By reading the review, one agrees of course with the initial choice of the interval, depending on three functions $F(r)$, $G(r)$, $H(r)$, but is soon led to wonder why Schwarzschild did abandon the polar coordinates r, ϑ, φ, that he had just introduced, and felt the need to go over to new spatial coordinates x_1, x_2, x_3, defined by the transformation:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \varphi.$$

One then wonders how Schwarzschild could determine his three new unknown functions f_1, $f_2 = f_3$, f_4 from Einstein's field equations without imposing one coordinate condition, that is not mentioned in Frank's account. Only by looking at the reviewed paper does one gather why Schwarzschild did work in that way. One discovers that he did not solve the field equations of the final version [11] of Einstein's theory, but the equations of the previous version, that Einstein had submitted [12] to the Prussian Academy of Sciences on November 11th, 1915. Those equations provided for the vacuum the same solutions as the final ones, but limited the covariance to unimodular coordinate transformations. They read:

$$(3) \qquad \sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0,$$

and

$$(4) \qquad |g_{\mu\nu}| = -1.$$

This very fact explains why Frank did not mention any coordinate condition in his review: Schwarzschild did not need one, for equation (4), meant by Einstein as a further field equation, played that rôle. This circumstance explains too the otherwise mysterious adoption of the coordinates x_1, x_2, x_3, named by Schwarzschild [1] "polar coordinates with determinant 1".

While the mentioned omission only hampers a proper understanding of Schwarzschild's procedure, the second omission had more far reaching consequences. It concerns the number of nontrivial constants of integration that occur in Schwarzschild's integration of (3) and (4). They are three, and were pointedly labeled as α, ρ and λ integration constants by the careful Schwarzschild, but only α appears in the final result reported by Frank. In the review, no word is spent about the existence of two more constants of integration, and about the way kept in fixing them. The omission is nearly irrelevant for λ, since Schwarzschild just set $\lambda = 1$ in order to fulfill both the Minkowskian boundary condition at infinity and the requirement that, for vanishing mass, the Minkowski manifold be retrieved. It is however crucial for ρ. The functions f_i, as they read before fixing ρ, are given in equations

(10)-(12) of the "Massenpunkt" paper [1]. They are:

$$f_1 = \frac{(3x_1 + \rho)^{-4/3}}{1 - \alpha(3x_1 + \rho)^{-1/3}},$$

(5)
$$f_2 = f_3 = (3x_1 + \rho)^{2/3},$$

$$f_4 = 1 - \alpha(3x_1 + \rho)^{-1/3}.$$

Schwarzschild notes that (5) satisfy all the conditions previously postulated for the solution, except for the condition of continuity, because f_1 is discontinuous when

(6) $$1 = \alpha(3x_1 + \rho)^{-1/3}, \quad i.e. \quad 3x_1 = \alpha^3 - \rho.$$

In order that this discontinuity coincides with the origin of x_1, namely, with the inner border of the manifold considered by him, Schwarzschild chose

(7) $$\rho = \alpha^3.$$

This is the reason why only the integration constant α survives in the final result reported by Frank. His review, however, by no means tells the reader that a problem, which had to become of fundamental importance for the future generations of relativists, had been seen by Schwarzschild, and deliberately solved in a certain way.

3. Hilbert's rederivation of the Schwarzschild solution

Frank's review of 1916 by its omissions certainly did not help in providing mathematicians and physicists alike with a clear idea both of the major issue that Schwarzschild had confronted when first solving Einstein's vacuum equations for the spherically symmetric, static case, and of the way out chosen by him. It was however Hilbert, with his revisitation [9] of the static, spherically symmetric problem, published in 1917, that definitely imposed the ostracism on the original Schwarzschild solution. He did so by attaching the name of Schwarzschild to the metric and the manifold defined by (1) and (2), that were instead the outcome of his own work, while dismissing in a footnote as "not advisable" the inequivalent, pondered choice of the manifold done by Schwarzschild. To document this occurrence, an English translation of the excerpt from Hilbert's paper [9] that deals with the rederivation of the spherically symmetric, static solution is reported in Appendix B; the above mentioned footnote is just at the end of the excerpt.

It must be aknowledged that, in this occasion, destiny exerted some of the irony of which it is capable with the great David Hilbert. In fact, as rightly noted [5] by Leonard Abrams, in the very paper by which he condemned Schwarzschild's deliberately chosen manifold to undeserved oblivion, Hilbert committed an error. A crucial constant of integration, that played in Hilbert's procedure just the rôle kept by ρ in Schwarzschild's calculation, was unknowingly allotted by him an arbitrary value, thereby fixing by pure chance the manifold in the "Schwarzschild" form (1) plus (2).

Hilbert's error was no doubt influential in rooting in many a relativist the wrong conviction that the manifold defined by (1) and (2) is a necessary outcome of the field equations of general relativity. Indeed, it corresponds just to one particular way of choosing the position of the inner border, that could have been adopted by Schwarzschild too, had he renounced his injunction of continuity for f_1, and chosen $\rho = 0$ instead of $\rho = \alpha^3$.

Let us consider Hilbert's derivation in some detail. In the footsteps of Einstein and Schwarzschild, he first postulated the conditions that the line element must obey, when written with respect to "Cartesian" coordinates, in order to describe a spherically symmetric, static manifold. Then he went over to polar coordinates and wrote the line element (42), where $F(r)$, $G(r)$ and $H(r)$ are three unknown functions. Due to the general covariance of the final field equations [11, 13] of the theory, that he himself had contributed to establish, in order to write a solution exempt from arbitrary functions, one must impose on the line element (42) one coordinate condition, that reduces the number of the unknown functions to two. Hilbert decided to fix $G(r)$ by introducing a new radial coordinate r^*, such that

$$r^* = \sqrt{G(r)}.$$

He then dropped the asterisk, thereby writing the line element (43), that contains only two unknown functions, $M(r)$ and $W(r)$, of the "new" r, and constitutes the canonical starting point for all the textbook derivations of the "Schwarzschild solution". This is quite legitimate. What is not legitimate, although first done by Hilbert and subsequently handed down to the posterity, is to assume without justification that the range of the "new" r is still $0 < r < \infty$, as it was for the "old" r, because this is tantamount to setting $\sqrt{G(0)} = 0$, an arbitrary choice [5], equivalent to setting $\rho = 0$ in Schwarzschild's result, reported in equation (5).

4. Forethoughts and afterthoughts

It might be asserted that Hilbert's error, when compared to Schwarzschild's meditated option for continuity, was a sort of *felix culpa*, because it was, perhaps by prophetic inspiration, fully in line with the subsequent understanding gained when the intrinsic viewpoint of differential geometry was correctly applied to general relativity. Through this improved understanding the discontinuity of f_1 occurring when $3x_1 = \alpha^3 - \rho$, that so much bothered Schwarzschild as to induce him to decide its relegation to the inner border of the manifold by setting $\rho = \alpha^3$, revealed itself to be a mere coordinate effect. On the contrary, the singularity occurring at $r = 0$ in, one should say, Hilbert's coordinates, revealed itself to be a genuine singularity of the manifold, defined in an invariant, local and intrinsic way through the pure knowledge of the metric. These facts are testified in any modern textbook by the exhibition of the polinomial invariants built with the metric, with the Riemann tensor and its covariant derivatives. Therefore one might think that while Hilbert, thanks to his error, stumbled over the

right manifold, Schwarzschild's conscious choice led him astray, due to the rudimentary status in which the differential geometry of his times was still lying. However, it will be noticed here that, despite the generally accepted opinion reported above, Hilbert's manifold appears to be afflicted with two defects, that are absent in Schwarzschild's manifold.

One of them was first taken care of by Synge, when he built from Hilbert's manifold a clever geometric construction [14], later mimicked by Kruskal and Szekeres in their maximal extensions [15, 16], in which the defect is no longer apparent. The shortcoming was later explained by Rindler [17, 18] to be at the origin of the strange duplication that the maximal extensions exhibit, with their bifurcate horizon and the unphysical prediction of the necessary coexistence of both a future and a past singularity.

The defect is simply told: Hilbert's manifold intrinsically disallows a consistent drawing of the time arrow; only the change of topology induced by either the Synge or by the Kruskal-Szekeres transformation with the inherent redoubling allows one to get a manifold where the arrow of time can be drawn without contradiction, in keeping with Synge's postulates [14].

A second defect of Hilbert's manifold is revealed [19, 20] with the contention that an invariant, local, intrinsic singularity is found at Schwarzschild's two-surface, provided that one does not limit the search, in this algebraically special manifold, to the singularities exhibited by the invariants build with the metric and with the Riemann tensor. In Schwarzschild's manifold, and in the $r > 2m$ part of Hilbert's manifold as well, through any event one can draw a unique path of absolute rest, because at each event the Killing equations

$$(8) \qquad\qquad \xi_{i;k} + \xi_{k;i} = 0,$$

and the condition of hypersurface orthogonality,

$$(9) \qquad\qquad \xi_{[i}\xi_{k,l]} = 0,$$

determine a unique timelike Killing vector ξ_i, that therefore uniquely identifies the direction of time. From ξ_i one can define the four-velocity

$$u^i \equiv \frac{\xi^i}{(\xi_i\xi^i)^{1/2}},$$

the four-acceleration

$$a^i \equiv \frac{\mathrm{D}u^i}{\mathrm{d}s}$$

as absolute derivative of u^i along its own direction, and the norm of this four-acceleration

$$(10) \qquad\qquad \alpha = (-a_i a^i)^{1/2}.$$

By using, say, Hilbert's manifold and coordinates, α comes to read

$$(11) \qquad\qquad \alpha = \frac{m}{r^{3/2}(r - 2m)^{1/2}}.$$

Hence it diverges in the limit when $r \to 2m$. Is not this divergence an invariant, local, intrinsic singularity on the inner border of Schwarzschild manifold, but, alas, in the interior of the Hilbert and of the Kruskal manifolds?

5. Appendix A: Frank's review of Schwarzschild's "Massenpunkt" paper

For arbitrary gravitational fields, the Author deals with the problem solved by *Einstein* for weak fields. He looks for a solution of the field equation satisfying the conditions that all the g_{ik} be independent of x_4, that $g_{14} = g_{24} = g_{34} = 0$, that the solution be spherically symmetric, and that the field vanish at infinity. Then, in polar coordinates, the line element must have the form

$$\mathrm{d}s^2 = F\mathrm{d}t^2 - (G + Hr^2)\mathrm{d}r^2 - Gr^2(\mathrm{d}\vartheta^2 + \sin^2 \vartheta \mathrm{d}\varphi^2),$$

where F, G, H are functions of r. If one poses

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \varphi,$$

it will be

$$\mathrm{d}s^2 = f_4\mathrm{d}x_4^2 - f_1\mathrm{d}x_1^2 - f_2\frac{\mathrm{d}x_2^2}{1 - x_2^2} - f_3\mathrm{d}x_3^2(1 - x_2^2),$$

where

$$f_2 = f_3, \quad f_4 = F, \quad f_1 = \frac{G}{r^4} + \frac{H}{r^2}, \quad f_2 = Gr^2.$$

Then, through integration of the field equations, it results

$$f_1 = \frac{1}{R^4}\frac{1}{1 - \frac{\alpha}{R}}, \quad f_2 = R^2, \quad f_4 = 1 - \frac{\alpha}{R},$$

where $R = \sqrt[3]{r^3 + \alpha^3}$ and α is a constant, that depends on the mass of the point. Therefore it is:

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{R}\right)\mathrm{d}t^2 - \frac{\mathrm{d}R^2}{1 - \frac{\alpha}{R}} - R^2(\mathrm{d}\vartheta^2 + \sin^2 \vartheta \mathrm{d}\varphi^2).$$

For the equations of motion it results as first integral:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2 = \frac{1 - h}{c^2} + \frac{h\alpha}{c^2}x - x^2 + \alpha x^3,$$

where $x = \frac{1}{R}$, and h is a constant of integration. If we substitute for R its approximate value r, from this equation the one found by *Einstein* is obtained, from which it results the motion of the perihelion of Mercury. If by n we mean the angular velocity of revolution, according to the exact solution the third *Kepler*'s law reads:

$$n^2 = \frac{\alpha}{2(r^3 + \alpha^3)}.$$

The proportionality between n^2 and r^{-3} therefore does not hold exactly; n does not grow without limit for decreasing r, but approaches itself to the maximal value $\frac{1}{\alpha\sqrt{2}}$.

6. Appendix B: Hilbert's derivation of the "Schwarzschild" metric

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The integration of the partial differential equations (36) is possible also in another case, that for the first time has been dealt with by Einstein[2] and by Schwarzschild[3]. In the following I provide for this case a way of solution that does not make any hypothesis on the gravitational potentials $g_{\mu\nu}$ at infinity, and that moreover offers advantages also for my further investigations. The hypotheses on the $g_{\mu\nu}$ are the following:

(1) The interval is referred to a Gaussian coordinate system - however g_{44} will still be left arbitrary; *i.e.* it is
$$g_{14} = 0, \quad g_{24} = 0, \quad g_{34} = 0.$$
(2) The $g_{\mu\nu}$ are independent of the time coordinate x_4.
(3) The gravitation $g_{\mu\nu}$ has central symmetry with respect to the origin of the coordinates.

According to Schwarzschild, if one poses
$$w_1 = r\cos\vartheta$$
$$w_2 = r\sin\vartheta\cos\varphi$$
$$w_3 = r\sin\vartheta\sin\varphi$$
$$w_4 = l$$

the most general interval corresponding to these hypotheses is represented in spatial polar coordinates by the expression

(42) $$F(r)\mathrm{d}r^2 + G(r)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) + H(r)\mathrm{d}l^2,$$

where $F(r)$, $G(r)$, $H(r)$ are still arbitrary functions of r. If we pose
$$r^* = \sqrt{G(r)},$$

we are equally authorised to interpret r^*, ϑ, φ as spatial polar coordinates. If we substitute in (42) r^* for r and then drop the symbol $*$, it results the expression

(43) $$M(r)\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2 + W(r)\mathrm{d}l^2,$$

where $M(r)$, $W(r)$ mean the two essentially arbitrary functions of r. The question is how the latter shall be determined in the most general way, so that the differential equations (36) happen to be satisfied.

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[2]Perihelbewegung des Merkur, Sitzungsber. d. Akad. zu Berlin. 1915, 831.

[3]Über das Gravitationsfeld eines Massenpunktes, Sitzunsber. d. Akad. zu Berlin. 1916, 189.

To this end the known expressions $K_{\mu\nu}$, K, given in my first communication, shall be calculated. The first step of this task consists in writing the differential equations of the geodesic line through variation of the integral

$$\int \left(M \left(\frac{\mathrm{d}r}{\mathrm{d}p} \right)^2 + r^2 \left(\frac{\mathrm{d}\vartheta}{\mathrm{d}p} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}p} \right)^2 + W \left(\left(\frac{\mathrm{d}l}{\mathrm{d}p} \right)^2 \right) \right) \mathrm{d}p.$$

We get as Lagrange equations:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}p^2} + \frac{M'}{2M} \left(\frac{\mathrm{d}r}{\mathrm{d}p} \right)^2 - \frac{r}{M} \left[\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}p} \right)^2 + \sin^2 \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}p} \right)^2 \right] - \frac{W'}{2M} \left(\frac{\mathrm{d}l}{\mathrm{d}p} \right)^2 = 0,$$

$$\frac{\mathrm{d}^2 \vartheta}{\mathrm{d}p^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}p} \frac{\mathrm{d}\vartheta}{\mathrm{d}p} - \sin \vartheta \cos \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}p} \right)^2 = 0,$$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}p^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}p} \frac{\mathrm{d}\varphi}{\mathrm{d}p} + 2 \cot \vartheta \frac{\mathrm{d}\vartheta}{\mathrm{d}p} \frac{\mathrm{d}\varphi}{\mathrm{d}p} = 0,$$

$$\frac{\mathrm{d}^2 l}{\mathrm{d}p^2} + \frac{W'}{W} \frac{\mathrm{d}r}{\mathrm{d}p} \frac{\mathrm{d}l}{\mathrm{d}p} = 0;$$

here and in the following calculation the symbol $'$ means differentiation with respect to r. By comparison with the general differential equations of the geodesic line:

$$\frac{\mathrm{d}^2 w_s}{\mathrm{d}p^2} + \sum_{\mu\nu} \{ {}^{\mu}_{s}{}^{\nu} \} \frac{\mathrm{d}w_\mu}{\mathrm{d}p} \frac{\mathrm{d}w_\nu}{\mathrm{d}p} = 0$$

we infer for the bracket symbols $\{ {}^{\mu}_{s}{}^{\nu} \}$ the following values (the vanishing ones are omitted):

$$\{ {}^1_1{}^1 \} = \frac{1}{2} \frac{M'}{M}, \quad \{ {}^2_1{}^2 \} = -\frac{r}{M}, \quad \{ {}^3_1{}^3 \} = -\frac{r}{M} \sin^2 \vartheta,$$

$$\{ {}^4_1{}^4 \} = -\frac{1}{2} \frac{W'}{M}, \quad \{ {}^1_2{}^2 \} = \frac{1}{r}, \quad \{ {}^3_2{}^3 \} = -\sin \vartheta \cos \vartheta,$$

$$\{ {}^1_3{}^3 \} = \frac{1}{r}, \quad \{ {}^2_3{}^3 \} = \cot \vartheta, \quad \{ {}^1_4{}^4 \} = \frac{1}{2} \frac{W'}{W}.$$

With them we form:

$$K_{11} = \frac{\partial}{\partial r} \left(\{ {}^1_1{}^1 \} + \{ {}^1_2{}^2 \} + \{ {}^1_3{}^3 \} + \{ {}^1_4{}^4 \} \right) - \frac{\partial}{\partial r} \{ {}^1_1{}^1 \}$$

$$+ \{ {}^1_1{}^1 \} \{ {}^1_1{}^1 \} + \{ {}^1_2{}^2 \} \{ {}^2_1{}^2 \} + \{ {}^1_3{}^3 \} \{ {}^3_1{}^3 \} + \{ {}^1_4{}^4 \} \{ {}^4_1{}^4 \}$$

$$- \{ {}^1_1{}^1 \} \left(\{ {}^1_1{}^1 \} + \{ {}^1_2{}^2 \} + \{ {}^1_3{}^3 \} + \{ {}^1_4{}^4 \} \right)$$

$$= \frac{1}{2} \frac{W''}{W} + \frac{1}{4} \frac{W'^2}{W^2} - \frac{M'}{rM} - \frac{1}{4} \frac{M'W'}{MW}$$

$$K_{22} = \frac{\partial}{\partial \vartheta} \{ {}^2_3{}^3 \} - \frac{\partial}{\partial r} \{ {}^2_1{}^2 \}$$

$$+ \{ {}^2_2{}^1 \} \{ {}^2_1{}^2 \} + \{ {}^2_1{}^2 \} \{ {}^1_2{}^2 \} + \{ {}^2_3{}^3 \} \{ {}^3_3{}^2 \}$$

$$-\left\{{}^2_1{}^2\right\}\left(\left\{{}^1_1{}^1\right\}+\left\{{}^1_2{}^2\right\}+\left\{{}^1_3{}^3\right\}+\left\{{}^1_4{}^4\right\}\right)$$

$$=-1-\frac{1}{2}\frac{rM'}{M^2}+\frac{1}{M}+\frac{1}{2}\frac{rW'}{MW}$$

$$K_{33}=-\frac{\partial}{\partial r}\left\{{}^3_1{}^3\right\}-\frac{\partial}{\partial\vartheta}\left\{{}^3_2{}^3\right\}$$

$$+\left\{{}^3_3{}^1\right\}\left\{{}^3_1{}^3\right\}+\left\{{}^3_3{}^2\right\}\left\{{}^3_2{}^3\right\}+\left\{{}^3_1{}^3\right\}\left\{{}^1_3{}^3\right\}+\left\{{}^3_2{}^3\right\}\left\{{}^2_3{}^3\right\}$$

$$-\left\{{}^3_1{}^3\right\}\left(\left\{{}^1_1{}^1\right\}+\left\{{}^1_2{}^2\right\}+\left\{{}^1_3{}^3\right\}+\left\{{}^1_4{}^4\right\}\right)-\left\{{}^3_2{}^3\right\}\left\{{}^2_3{}^3\right\}$$

$$=\sin^2\vartheta\left(-1-\frac{1}{2}\frac{rM'}{M^2}+\frac{1}{M}+\frac{1}{2}\frac{rW'}{MW}\right)$$

$$K_{44}=-\frac{\partial}{\partial r}\left\{{}^4_1{}^4\right\}+\left\{{}^4_4{}^1\right\}\left\{{}^4_1{}^4\right\}+\left\{{}^4_1{}^4\right\}\left\{{}^4_4{}^1\right\}$$

$$-\left\{{}^4_1{}^4\right\}\left(\left\{{}^1_1{}^1\right\}+\left\{{}^1_2{}^2\right\}+\left\{{}^1_3{}^3\right\}+\left\{{}^1_4{}^4\right\}\right)$$

$$=\frac{1}{2}\frac{W''}{M}-\frac{1}{4}\frac{M'W'}{M^2}-\frac{1}{4}\frac{W'^2}{MW}+\frac{W'}{rM}$$

$$K=\sum_s g^{ss}K_{ss}=\frac{W''}{MW}-\frac{1}{2}\frac{W'^2}{MW^2}-2\frac{M'}{rM^2}-\frac{1}{2}\frac{M'W'}{M^2W}-\frac{2}{r^2}+\frac{2}{r^2M}+2\frac{W'}{rMW}.$$

Since

$$\sqrt{g}=\sqrt{MW}\,r^2\sin\vartheta$$

it is found

$$K\sqrt{g}=\left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)'-2\frac{rM'\sqrt{W}}{M^{\frac{3}{2}}}-2\sqrt{MW}+2\sqrt{\frac{W}{M}}\right\}\sin\vartheta$$

and, if we set

$$M=\frac{r}{r-m},\qquad W=w^2\frac{r-m}{r},$$

where henceforth m and w become the unknown functions of r, we eventually obtain

$$K\sqrt{g}=\left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)'-2wm'\right\}\sin\vartheta.$$

Therefore the variation of the quadruple integral

$$\int\int\int\int K\sqrt{g}\,\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi\mathrm{d}l$$

is equivalent to the variation of the single integral

$$\int wm'\mathrm{d}r$$

and leads to the Lagrange equations

(44)
$$m'=0,$$
$$w'=0.$$

One easily satisfies oneself that these equations effectively entail the vanishing of all the $K_{\mu\nu}$; they represent therefore essentially the most general solution of the equations (36) under the hypotheses (1), (2), (3) previously made. If we take as integrals of (44) $m = \alpha$, where α is a constant, and $w = 1$ (a choice that evidently does not entail any essential restriction) from (43) with $l = it$ it results the looked for interval in the form first found by Schwarzschild

$$(45) \qquad G(\mathrm{d}r, \mathrm{d}\vartheta, \mathrm{d}\varphi, \mathrm{d}l) = \frac{r}{r - \alpha}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2 \sin^2 \vartheta \mathrm{d}\varphi^2 - \frac{r - \alpha}{r}\mathrm{d}t^2.$$

The singularity of this interval for $r = 0$ vanishes only when it is assumed $\alpha = 0$, *i.e.*: under the hypotheses (1), (2), (3) the interval of the pseudo-Euclidean geometry is the only regular interval that corresponds to a world without electricity.

For $\alpha \neq 0$, $r = 0$ and, with positive values of α, also $r = \alpha$ happen to be such points that in them the interval is not regular. I call an interval or a gravitational field $g_{\mu\nu}$ *regular* in a point if, through an invertible one to one transformation, it is possible to introduce a coordinate system such that for it the corresponding functions $g'_{\mu\nu}$ are regular in that point, *i.e.* in it and in its neighbourhood they are continuous and differentiable at will, and have a determinant g' different from zero.

Although in my opinion only regular solutions of the fundamental equations of physics immediately represent the reality, nevertheless just the solutions with non regular points are an important mathematical tool for approximating characteristic regular solutions - and in this sense, according to the procedure of Einstein and Schwarzschild, the interval (45), not regular for $r = 0$ and for $r = \alpha$, must be considered as expression of the gravitation of a mass distributed with central symmetry in the surroundings of the origin[4]. . . . .

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[4]Transforming to the origin the position $r = \alpha$, like Schwarzschild did, is in my opinion not advisable; moreover Schwarzschild's transformation is not the simplest one, that reaches this scope.

References

[1] Schwarzschild, K., (1916). Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl., 189 (submitted 13 Jan. 1916).

[2] Antoci, S., and Liebscher, D.-E. (2003). Gen. Relativ. Gravit. **35**, 945.

[3] English translation of [1]: (2003). Gen. Relativ. Gravit. **35**, 951. Also at the Cornell Archive: http://arXiv.org/abs/physics/9905030

[4] Abrams, L.S. (1979). Phys. Rev. D **20**, 2474.
 Cornell Archive: http://arXiv.org/abs/gr-qc/0201044

[5] Abrams, L.S. (1989). Can. J. Phys. **67**, 919.
 Cornell Archive: http://arXiv.org/abs/gr-qc/0102055

[6] Pekeris, C.L. (1982). Proc. Natl. Acad Sci. USA **79**, 6404.

[7] Frank, Ph. (1916). Jahrbuch Forts. Math. **46**, 1296.

[8] Droste, J. (1917). Proc. K. Ned. Akad. Wet., Ser. A **19**, 197 (submitted 27 May 1916). Reprinted (2002). Gen. Relativ. Gravit. **34**, 1545.

[9] Hilbert, D. (1917). Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl., 53 (submitted 23 Dec. 1916).

[10] Weyl, H. (1917). Ann. Phys. (Leipzig) **54**, 117.

[11] Einstein, A. (1915). Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl., 844 (submitted 25 Nov. 1915).

[12] Einstein, A. (1915). Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl., 778, 799 (submitted 11 Nov. 1915).

[13] Hilbert, D. (1915) Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl., 395 (submitted 20 Nov. 1915).

[14] Synge, J.L. (1950). Proc. R. Irish Acad. **53A**, 83.

[15] Kruskal, M.D., (1960). Phys. Rev. **119**, 1743.

[16] Szekeres, G., (1960). Publ. Math. Debrecen **7**, 285.

[17] Rindler, W., (1990). *Proceedings of the International Institute of Theoretical Physics, School on Relativity, Supersymmetry and Strings*, A. Rosenblum (ed.), Plenum Press, New York, p. 82-83.

[18] Rindler, W., (2001). *Relativity, special, general and cosmological*, Oxford University Press, Oxford, p. 266-267.

[19] Antoci, S., and Liebscher, D.-E., (2001). Astron. Nachr. **322**, 137.
 Cornell Archive: http://arXiv.org/abs/gr-qc/0102084

[20] Antoci, S., Liebscher, D.-E., and Mihich, L. (2001). Class. Quant. Grav. **18**, 3463.
 Cornell Archive: http://arXiv.org/abs/gr-qc/0104035

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