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THE FOUNDATIONS OF PHYSICS (SECOND COMMUNICATION)

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In my first communication[1] I proposed a system of basic equations of physics. Before turning to the theory of integrating these equations it seems necessary to discuss some more general questions of a logical as well as physical nature.

First we introduce in place of the world parameters w_s $(s = 1, 2, 3, 4)$ the most general *real* spacetime coordinates x_s $(s = 1, 2, 3, 4)$ by putting

$$w_1 = x_1, \qquad w_2 = x_2, \qquad w_3 = x_3, \qquad w_4 = x_4,$$

and correspondingly in place of

$$ig_{14}, \qquad ig_{24}, \qquad ig_{34}, \qquad -g_{44},$$

we write simply

$$g_{14}, \qquad g_{24}, \qquad g_{34}, \qquad g_{44}.$$

The new $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ —the gravitational potentials of Einstein—shall then all be real functions of the real variables x_s $(s = 1, 2, 3, 4)$ of such a type that, in the representation of the quadratic form

$$G(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu \tag{28}$$

as a sum of four squares of linear forms of the X_s, three squares always occur with positive sign, and one square with negative | sign: thus the quadratic form (28) provides our four dimensional world of the x_s with the metric of a pseudo-geometry. [54]
The determinant g of the $g_{\mu\nu}$ turns out to be negative.

1 This journal, 20 November 1915.

Jürgen Renn (ed.). *The Genesis of General Relativity,* Vol. 4
Gravitation in the Twilight of Classical Physics: The Promise of Mathematics.

If a curve

$$x_s = x_s(p) \qquad (s = 1, 2, 3, 4)$$

is given in this geometry, where $x_s(p)$ mean some arbitrary real functions of the parameter p, then it can be divided into pieces of curves on each of which the expression

$$G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right)$$

does not change sign: A piece of the curve for which

$$G\left(\frac{dx_s}{dp}\right) > 0$$

shall be called a *segment* and the integral along this piece of curve

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)}dp$$

shall be the *length of the segment*; a piece of the curve for which

$$G\left(\frac{dx_s}{dp}\right) < 0$$

will be called a *time line*, and the integral

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)}dp$$

evaluated along this piece of curve shall be the *proper time of the time line*; finally a piece of curve along which

$$G\left(\frac{dx_s}{dp}\right) = 0$$

shall be called a *null line*.

To visualize these concepts of our pseudo geometry we imagine two ideal measuring devices: the *measuring thread* by means of which we are able to measure the length λ of any segment, and secondly the *light clock* with which we can determine the proper time of any time line. The thread shows zero and the light clock stops along every null line, whereas the former fails totally along a time line, and the latter along a segment. |

[55] First we show that each of the two instruments suffices to compute with its aid the values of the $g_{\mu\nu}$ as functions of x_s, as soon as a definite spacetime coordinate system x_s has been introduced. Indeed we choose any set of 10 segments, which all converge on the same world point x_s, from different directions, so that this endpoint

assumes the same parameter value p on each. At this end point we have the equation, for each of the 10 segments,

$$\left(\frac{d\lambda^{(h)}}{dp}\right)^2 = G\left(\frac{dx_s^{(h)}}{dp}\right), \qquad (h = 1, 2, \ldots, 10);$$

here the left-hand sides are known as soon as we have determined the lengths $\lambda^{(h)}$ by means of the thread. We introduce the abbreviations

$$D(u) = \begin{vmatrix} \left(\frac{dx_1^{(1)}}{dp}\right)^2, & \frac{dx_1^{(1)}}{dp}\frac{dx_2^{(1)}}{dp}, & \ldots, & \left(\frac{dx_4^{(1)}}{dp}\right)^2, & \left(\frac{d\lambda^{(1)}}{dp}\right)^2 \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \left(\frac{dx_1^{(10)}}{dp}\right)^2, & \frac{dx_1^{(10)}}{dp}\frac{dx_2^{(10)}}{dp}, & \ldots, & \left(\frac{dx_4^{(10)}}{dp}\right)^2, & \left(\frac{d\lambda^{(10)}}{dp}\right)^2 \\ X_1^2, & X_1X_2, & \ldots, & X_4^2, & u \end{vmatrix},$$

so that clearly

$$G(X_s) = -\frac{D(0)}{\frac{\partial D}{\partial u}}, \tag{29}$$

whereby also the condition on the directions of the chosen 10 segments at the point $x_s(p)$

$$\frac{\partial D}{\partial u} \neq 0$$

is seen to be necessary.

When G has been calculated according to (29), the use of this procedure for any 11th segment ending at $x_s(p)$ would yield the equation

$$\left(\frac{d\lambda^{(11)}}{dp}\right)^2 = G\left(\frac{dx_s^{(11)}}{dp}\right),$$

and this equation would then both verify the correctness of the instrument and confirm experimentally that the postulates of the theory apply to the real world.

Corresponding reasoning applies to the light clock. |

The axiomatic construction of our pseudo-geometry could be carried out without difficulty: first an axiom should be established from which it follows that length resp. proper time must be integrals whose integrand is only a function of the x_s and their first derivatives with respect to the parameter; suitable for such an axiom would be the property of development of the thread or the well-known envelope theorem for geodesic lines. Secondly an axiom is needed whereby the theorems of the pseudo-Euclidean geometry, that is the old principle of relativity, shall be valid in infinitesi- [56]

mal regions; for this the axiom put down by W. Blaschke[2] would be particularly suitable, which states that the condition of orthogonality for any two directions—segments or time lines—shall always be a symmetric relation.

Let us briefly summarize the main facts that the Monge-Hamilton theory of differential equations teaches us for our pseudo-geometry.

With every world point x_s there is associated a cone of second order, with vertex at x_s, and determined in the running point coordinates X_s by the equation

$$G(X_1 - x_1, X_2 - x_2, X_3 - x_3, X_4 - x_4) = 0;$$

this shall be called the *null cone* belonging to the point x_s. The totality of null cones form a four dimensional field of cones, which is associated on the one hand with "Monge's" differential equation

$$G(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}) = 0,$$

and on the other hand with "Hamilton's" partial differential equation

$$H(\frac{df}{dx_1}, \frac{df}{dx_2}, \frac{df}{dx_3}, \frac{df}{dx_4}) = 0, \tag{30}$$

where H denotes the quadratic form

$$H(U_1, U_2, U_3, U_4) = \sum_{\mu\nu} g^{\mu\nu} U_\mu U_\nu$$

reciprocal to G. The characteristics of Monge's and at the same time those of Hamilton's partial differential equation (30) are the geodesic null lines. All the geodesic null lines originating at one particular world point a_s $(s = 1, 2, 3, 4)$ generate a three
[57] dimensional point manifold, which | shall be called the *time divide* belonging to the world point a_s. This divide has a node at a_s, whose tangent cone is precisely the null cone belonging to a_s. If we transform the equation of the time divide into the form

$$x_4 = \varphi(x_1, x_2, x_3),$$

then

$$f = x_4 - \varphi(x_1, x_2, x_3)$$

is an integral of Hamilton's differential equation (30). All the time lines originating at the point a_s remain totally in the interior of that four dimensional part of the world whose boundary is the time divide of a_s.

After these preparations we turn to the problem of *causality* in the new physics.

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2 "Räumliche Variationsprobleme mit symmetrischer Transversalitätsbedingung." *Leipziger Berichte, Math.-phys. Kl.* 68 (1916) p. 50.

Up to now all coordinate systems x_s, that result from any one by arbitrary transformation have been regarded as equally valid. This arbitrariness must be restricted when we want to realize the concept that two world points on the same time line can be related as cause and effect, and that it should then no longer be possible to transform such world points to be simultaneous. In declaring x_4 as the *true* time coordinate we adopt the following definition:

A *true* spacetime coordinate system is one for which the following four inequalities hold, in addition to $g < 0$:

$$g_{11} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0, \qquad g_{44} < 0. \tag{31}$$

A transformation that transforms one such spacetime coordinate system into another true spacetime coordinate system shall be called a *true* spacetime coordinate transformation.

The four inequalities mean that at any world point a_s the associated null cone excludes the linear space

$$x_4 = a_4,$$

but contains in its interior the line

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3;$$

the latter line is therefore always a time line. |

Let any time line $x_s = x_s(p)$ be given; because [58]

$$G\left(\frac{dx_s}{dp}\right) < 0$$

it follows that in a true spacetime coordinate system we must always have

$$\frac{dx_4}{dp} \neq 0,$$

and therefore that along a time line the true time coordinate x_4 must always increase resp. decrease. Because a time line remains a time line upon every coordinate transformation, therefore two world points along one time line can never be given the same value of the time coordinate x_4 through a true spacetime transformation; that is, they cannot be transformed to be simultaneous.

On the other hand, if the points of a curve can be truly transformed to be simultaneous, then after this transformation we have for this curve

$$x_4 = \text{const.}, \qquad \text{that is} \qquad \frac{dx_4}{dp} = 0,$$

therefore

$$G\left(\frac{dx_s}{dp}\right) = \sum_{\mu\nu} g_{\mu\nu}\frac{dx_\mu}{dp}\frac{dx_\nu}{dp}, \qquad (\mu, \nu = 1, 2, 3),$$

and here the right side is positive because of the first three of our inequalities (31); the curve therefore characterizes a *segment*.

So we see that the concepts of cause and effect, which underlie the principle of causality, also do not lead to any inner contradictions whatever in the new physics, if we only take the inequalities (31) always to be part of our basic equations, that is if we confine ourselves to using *true* spacetime coordinates.

At this point let us take note of a special spacetime coordinate system that will later be useful and which I will call the *Gaussian coordinate system*, because it is the generalization of the system of geodesic polar coordinates introduced by Gauss in the theory of surfaces. In our four-dimensional world let any three-dimensional space be given so that every curve confined to that space is a segment: *a space of segments*, as
[59] I would like to call it; | let x_1, x_2, x_3 be any point coordinates of this space. We now construct at every point x_1, x_2, x_3 of this space the geodesic orthogonal to it, which will be a time line, and on this line we mark off x_4 as proper time; the point in the four-dimensional world so obtained is given coordinate values $x_1x_2x_3x_4$. In these coordinates we have, as is easily seen,

$$G(X_s) = \sum_{\mu\nu}^{1,2,3} g_{\mu\nu}X_\mu X_\nu - X_4^2, \tag{32}$$

that is, the Gaussian coordinate system is characterized analytically by the equations

$$g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0, \qquad g_{44} = 0. \tag{33}$$

Because of the nature of the three dimensional space $x_4 = 0$ we presupposed, the quadratic form on the right-hand side of (32) in the variables X_1, X_2, X_3 is necessarily positive definite, so the first three of the inequalities (31) are satisfied, and since this also applies to the fourth, the Gaussian coordinate system always turns out to be a *true* spacetime coordinate system.

We now return to the investigation of the principle of causality in physics. As its main contents we consider the fact, valid so far in every physical theory, that from a knowledge of the physical quantities and their time derivatives in the present the future values of these quantities can always be determined: without exception the laws of physics to date have been expressed in a system of differential equations in which the number of the functions occurring in them was essentially the same as the number of independent differential equations; and thus the well-known general Cauchy theorem on the existence of integrals of partial differential equations directly offered the rationale of proof for the above fact.

Now, as I emphasized particularly in my first communication, the basic equations of physics (4) and (5) established there are by no means of the type characterized above; rather, according to Theorem I, four of them are a consequence of the rest: we regarded the four Maxwell equations (5) as a consequence of the ten gravitational equations (4), and so we have for the 14 potentials $g_{\mu\nu}$, q_s only 10 equations (4) that are essentially independent of each other. |

As soon as we maintain the demand of general invariance for the basic equations of physics the circumstance just mentioned is essential and even necessary. Because if there were further invariant equations, independent of (4), for the 14 potentials, then introduction of a Gaussian coordinate system would lead for the 10 physical quantities as per (33), [60]

$$g_{\mu\nu} \quad (\mu, \nu = 1, 2, 3), \qquad q_s \quad (s = 1, 2, 3, 4)$$

to a system of equations that would again be mutually independent, and mutually contradictory, because there are more than 10 of them.

Under such circumstances then, as occur in the new physics of general relativity, it is by no means any longer possible from knowledge of physical quantities in present and past to derive uniquely their future values. To show this intuitively on an example, let our basic equations (4) and (5) of the first communication be integrated in the special case corresponding to the presence of a single electron permanently at rest, so that the 14 potentials

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x_1, x_2, x_3) \\ q_s &= q_s(x_1, x_2, x_3) \end{aligned}$$

become definite functions of x_1, x_2, x_3, all independent of the time x_4, and in addition such that the first three components r_1, r_2, r_3 of the four-current density vanish. Then we apply the following coordinate transformation to these potentials:

$$\begin{cases} x_1 = x'_1 & \text{for } x'_4 \leq 0 \\ x_1 = x'_1 + e^{-\frac{1}{x'^2_4}} & \text{for } x'_4 > 0 \end{cases}$$
$$\begin{aligned} x_2 &= x'_2 \\ x_3 &= x'_3 \\ x_4 &= x'_4. \end{aligned}$$

For $x'_4 \leq 0$ the transformed potentials $g'_{\mu\nu}$, q'_s are the same functions of x'_1, x'_2, x'_3 as the $g_{\mu\nu}$, q_s of the original variables x_1, x_2, x_3, whereas the $g'_{\mu\nu}$, q'_s for $x'_4 > 0$ depend in an essential way also on the time coordinate x'_4; that is, the potentials $g'_{\mu\nu}$, q'_s represent an electron that is at rest until $x'_4 = 0$, but then puts its components into motion. |

Nonetheless I believe that it is only necessary to formulate more sharply the idea on which the principle of general relativity[3] is based, in order to maintain the principle of causality also in the new physics. Namely, to follow the essence of the new rel- [61]

ativity principle we must demand invariance not only for the general laws of physics, but we must accord invariance to each separate statement in physics that is to have physical meaning—in accordance with this, that in the final analysis it must be possible to establish each physical fact by thread or light clock, that is, instruments of *invariant* character. In the theory of curves and surfaces, where a statement in a chosen parametrization of the curve or surface has no geometrical meaning for the curve or surface itself, if this statement does not remain invariant under any arbitrary transformation of the parameters or cannot be brought to invariant form; so also in physics we must characterize a statement that does not remain invariant under any arbitrary transformation of the coordinate system as *physically meaningless*. For example, in the case considered above of the electron at rest, the statement that, say at the time $x_4 = 1$ this electron is at rest, has no physical meaning because this statement is not invariant.

Concerning the principle of causality, let the physical quantities and their time derivatives be known at the present in some given coordinate system: then a statement will only have physical meaning if it is invariant under all those transformations, for which the coordinates just used for the present remain unchanged; I maintain that statements of this type for the future are all uniquely determined, that is, *the principle of causality holds in this form*:

From present knowledge of the 14 physical potentials $g_{\mu\nu}$, q_s all statements about them for the future follow necessarily and uniquely provided they are physically meaningful.

To prove this proposition we use the *Gaussian* spacetime coordinate system. Introducing (33) into the basic equations (4) of the first communication yields for the 10 potentials |

[62]
$$g_{\mu\nu} \quad (\mu, \nu = 1, 2, 3), \qquad q_s \quad (s = 1, 2, 3, 4) \tag{34}$$

a system of as many partial differential equations; if we integrate these on the basis of the given initial values at $x_4 = 0$, we find uniquely the values of (34) for $x_4 > 0$. Since the Gaussian coordinate system itself is uniquely determined, therefore also all statements about those potentials (34) with respect to these coordinates are of invariant character.

The forms, in which physically meaningful, i.e. invariant, statements can be expressed mathematically are of great variety.

First. This can be done by means of an invariant coordinate system. Like the Gaussian system used above one can apply the well-known Riemannian one, as well as that spacetime coordinate system in which electricity appears at rest with unit current density. As at the end of the first communication, let $f(q)$ denote the function occurring in Hamilton's principle and depending on the invariant

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3 In his original theory, now abandoned, A. Einstein (*Sitzungsberichte der Akad. zu Berlin*, 1914, p. 1067) had indeed postulated certain 4 non-invariant equations for the $g_{\mu\nu}$, in order to save the causality principle in its old form.

$$q = \sum_{kl} q_k q_l g^{kl},$$

then

$$r^s = \frac{\partial f(q)}{\partial q_s}$$

is the four-current density of electricity; it represents a contravariant vector and therefore can certainly be transformed to $(0, 0, 0, 1)$, as is easily seen. If this is done, then from the four equations

$$\frac{\partial f(q)}{\partial q_s} = 0 \quad (s = 1, 2, 3), \qquad \frac{\partial f(q)}{\partial q_4} = 1$$

the four components of the four-potential q_s can be expressed in terms of the $g_{\mu\nu}$, and every relation between the $g_{\mu\nu}$ in this or in one of the first two coordinate systems is then an invariant statement. For particular solutions of the basic equations there may be special invariant coordinate systems; for example, in the case treated below of the centrally symmetric gravitational field r, ϑ, φ, t form an invariant system of coordinates up to rotations.

Second. The statement, *according to which a coordinate system can be found* in which the 14 potentials $g_{\mu\nu}$, q_s have certain definite values in the future, or fulfill certain definite conditions, is always an invariant and therefore a physically meaningful one. The mathematically invariant expression for | such a statement is obtained by [63] eliminating the coordinates from those relations. The case considered above, of the electron at rest, provides an example: the essential and physically meaningful content of the causality principle is here expressed by the statement that the electron which is at rest for the time $x_4 \leq 0$ will, *for a suitably chosen spacetime coordinate system*, also remain at rest in all its parts for the future $x_4 > 0$.

Third. A statement is also invariant and thus has physical meaning if it is supposed to be valid in any arbitrary coordinate system. An example of this are Einstein's energy-momentum equations having divergence character. For, although Einstein's energy does not have the property of invariance, and the differential equations he put down for its components are by no means covariant as a system of equations, nevertheless the assertion contained in them, that they shall be satisfied in any coordinate system, is an invariant demand and therefore it carries physical meaning.

According to my exposition, physics is a four-dimensional pseudo-geometry, whose metric $g_{\mu\nu}$ is connected to the electromagnetic quantities, i.e. to the matter, by the basic equations (4) and (5) of my first communication. With this understanding, an old geometrical question becomes ripe for solution, namely whether and in what sense Euclidean geometry—about which we know from mathematics only that it is a logical structure free from contradictions—also possesses validity in the real world.

The old physics with the concept of absolute time took over the theorems of Euclidean geometry and without question put them at the basis of every physical theory. Gauss as well proceeded hardly differently: he constructed a hypothetical non-

Euclidean physics, by maintaining the absolute time and revoking only the parallel axiom from the propositions of Euclidean geometry; a measurement of the angles of a triangle of large dimensions showed him the invalidity of this non-Euclidean physics.

The new physics of Einstein's principle of general relativity takes a totally different position vis-à-vis geometry. It takes neither Euclid's nor any other particular geometry *a priori* as basic, in order to deduce from it the proper laws of physics, but,
[64] as I showed in my first communication, | the new physics provides at one fell swoop through one and the same Hamilton's principle the geometrical and the physical laws, namely the basic equations (4) and (5), which tell us how the metric $g_{\mu\nu}$—at the same time the mathematical expression of the phenomenon of gravitation—is connected with the values q_s of the electrodynamic potentials.

Euclidean geometry is *an action-at-a-distance law foreign to the modern physics*: By revoking the Euclidean geometry as a general presupposition of physics, the theory of relativity maintains instead that geometry and physics have identical character and are based as *one* science on a common foundation.

The geometrical question mentioned above amounts to the investigation, whether and under what conditions the four-dimensional Euclidean pseudo-geometry

$$\begin{array}{llll} g_{11} = 1, & g_{22} = 1, & g_{33} = 1, & g_{44} = -1 \\ g_{\mu\nu} = 0 & (\mu \neq \nu) & & \end{array} \tag{35}$$

is a solution, or even the only regular solution, of the basic physical equations.

The basic equations (4) of my first communication are, due to the assumption (20) made there:

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0,$$

where

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\Big(K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}\Big).$$

When the values (35) are substituted, we have

$$[\sqrt{g}K]_{\mu\nu} = 0 \tag{36}$$

and for

$$q_s = 0 \qquad (s = 1, 2, 3, 4)$$

we have

$$\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0;$$

that is, when all electricity is removed, the pseudo-Euclidean geometry is possible. The question whether it is also necessary in this case, i.e. whether—or under certain additional conditions—the values (35), and those values of the $g_{\mu\nu}$ resulting from coordinate transformation of the latter, are the only regular solutions of the equations (36) is a mathematical problem not to be discussed here in general. Instead I confine myself | to presenting some thoughts concerning this problem in particular. [65]

For this we return to the original world coordinates of my first communication

$$w_1 = x_1, \qquad w_2 = x_2, \qquad w_3 = x_3, \qquad w_4 = ix_4,$$

and give the corresponding meaning to the $g_{\mu\nu}$.

In the case of the pseudo-Euclidean geometry we have

$$g_{\mu\nu} = \delta_{\mu\nu},$$

where

$$\delta_{\mu\nu} = 1, \qquad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

For every metric in the neighborhood of this pseudo-Euclidean geometry the ansatz

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \ldots \tag{37}$$

is valid, where ε is a quantity converging to zero, and $h_{\mu\nu}$ are functions of the w_s. I make the following two assumptions about the metric (37):

I. The $h_{\mu\nu}$ shall be independent of the variable w_4.

II. The $h_{\mu\nu}$ shall show a certain regular behavior at infinity.

Now, if the metric (37) is to satisfy the differential equation (36) for all ε then it follows that the $h_{\mu\nu}$ must necessarily satisfy certain linear homogeneous partial differential equations of second order. If we substitute, following Einstein[4]

$$h_{\mu\nu} = k_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_s k_{ss}, \qquad (k_{\mu\nu} = k_{\nu\mu}) \tag{38}$$

and assume among the 10 functions $k_{\mu\nu}$ the four relations

$$\sum_s \frac{dk_{\mu s}}{dw_s} = 0, \qquad (\mu = 1, 2, 3, 4) \tag{39}$$

then these differential equations become:

$$\Box k_{\mu\nu} = 0, \tag{40}$$

where the abbreviation

4 "Näherungsweise Integration der Feldgleichungen der Gravitation." *Berichte d. Akad. zu Berlin* 1916, p. 688.

$$\square = \sum_s \frac{\partial^2}{\partial w_s^2}$$

has been used.

Because of the ansatz (38) the relations (39) are restrictive assumptions for the
[66] functions $h_{\mu\nu}$; however I will | show how one can always achieve, by suitable infinitesimal transformation of the variables w_1, w_2, w_3, w_4, that those restrictive assumptions are satisfied for the corresponding functions $h'_{\mu\nu}$ after the transformation.

To this end one should determine four functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, which satisfy respectively the differential equations

$$\square\varphi_\mu = \frac{1}{2}\frac{\partial}{\partial w_\mu}\sum_\nu h_{\nu\nu} - \sum_\nu \frac{\partial h_{\mu\nu}}{\partial w_\nu}. \tag{41}$$

By means of the infinitesimal transformation

$$w_s = w'_s + \varepsilon\varphi_s,$$

$g_{\mu\nu}$ becomes

$$g'_{\mu\nu} = g_{\mu\nu} + \varepsilon\sum_\alpha g_{\alpha\nu}\frac{\partial\varphi_\alpha}{\partial w_\mu} + \varepsilon\sum_\alpha g_{\alpha\mu}\frac{\partial\varphi_\alpha}{\partial w_\nu} + \ldots$$

or because of (37) it becomes

$$g'_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h'_{\mu\nu} + \ldots,$$

where I have put

$$h'_{\mu\nu} = h_{\mu\nu} + \frac{\partial\varphi_\nu}{\partial w_\mu} + \frac{\partial\varphi_\mu}{\partial w_\nu}.$$

If we now choose

$$k_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_s h'_{ss},$$

then these functions satisfy Einstein's condition (39) because of (41), and we have

$$h'_{\mu\nu} = k_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_s k_{ss} \qquad (k_{\mu\nu} = k_{\nu\mu}).$$

The differential equations (40), which must be valid according to the above argument for the $k_{\mu\nu}$ we found, become due to assumption I

$$\frac{\partial^2 k_{\mu\nu}}{\partial w_1^2} + \frac{\partial^2 k_{\mu\nu}}{\partial w_2^2} + \frac{\partial^2 k_{\mu\nu}}{\partial w_3^2} = 0,$$

and, since assumption II—*mutatis mutandis*—allows the conclusion that the $k_{\mu\nu}$ approach constants at infinity, it follows that these must be constant in general, that is: *By varying the metric of the pseudo-Euclidean geometry under the assumptions I and II it is not possible to obtain a regular metric that is not likewise pseudo-Euclidean and which also corresponds to a world free of electricity.* |

The integration of the partial differential equations (36) can be performed in yet another case, first treated by Einstein[5] and by Schwarzschild.[6] In the following I present for this case a procedure that makes no assumptions about the gravitational potentials $g_{\mu\nu}$ at infinity, and which moreover offers advantages for my later investigations. The assumptions about the $g_{\mu\nu}$ are the following: [67]

1. The metric is represented in a Gaussian coordinate system, except that g_{44} is left arbitrary, i.e. we have

$$g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0.$$

2. The $g_{\mu\nu}$ are independent of the time coordinate x_4.
3. The gravitation $g_{\mu\nu}$ is centrally symmetric with respect to the origin of coordinates.

According to Schwarzschild the most general metric conforming to these assumptions is represented in polar coordinates, where

$$\begin{aligned} w_1 &= r\cos\vartheta \\ w_2 &= r\sin\vartheta\cos\varphi \\ w_3 &= r\sin\vartheta\sin\varphi \\ w_4 &= l, \end{aligned}$$

by the expression

$$F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2 \tag{42}$$

where $F(r), G(r), H(r)$ are still arbitrary functions of r. If we put

$$r^* = \sqrt{G(r)},$$

then we are equally justified in interpreting r^*, ϑ, φ as spatial polar coordinates. If we introduce r^* in (42) instead of r and then eliminate the sign $*$, the result is the expression

$$M(r)dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 + W(r)dl^2, \tag{43}$$

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5 "Perihelbewegung des Merkur." *Situngsber. d. Akad. zu Berlin*. 1915, p. 831.

6 "Über das Gravitationsfeld eines Massenpunktes." *Sitzungsber. d. Akad. zu Berlin*. 1916, p. 189.

where $M(r)$, $W(r)$ mean the two essential, arbitrary functions of r. The question is whether and how these can be determined in the most general way so that the differential equations (36) enjoy satisfaction. |

[68] To this end the well-known expressions $K_{\mu\nu}$, K given in my first communication must be calculated. The first step in this is the derivation of the differential equations for geodesic lines by variation of the integral

$$\int\left(M\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\vartheta}{dp}\right)^2 + r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2 + W\left(\frac{dl}{dp}\right)^2\right)dp.$$

As Lagrange equations we obtain these:

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{M'}{M}\left(\frac{dr}{dp}\right)^2 - \frac{r}{M}\left(\frac{d\vartheta}{dp}\right)^2 - \frac{r}{M}\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2 - \frac{1}{2}\frac{W'}{M}\left(\frac{dl}{dp}\right)^2 = 0,$$

$$\frac{d^2\vartheta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\vartheta}{dp} - \sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dp}\right)^2 = 0,$$

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} + 2\cot\vartheta\frac{d\vartheta}{dp}\frac{d\varphi}{dp} = 0,$$

$$\frac{d^2l}{dp^2} + \frac{W'}{W}\frac{dr}{dp}\frac{dl}{dp} = 0;$$

here and in the following calculation the sign $'$ denotes the derivative with respect to r. By comparison with the general differential equations of geodesic lines:

$$\frac{d^2w_s}{dp^2} + \sum_{\mu\nu}\begin{Bmatrix}\mu\nu\\ s\end{Bmatrix}\frac{dw_\mu}{dp}\frac{dw_\nu}{dp} = 0,$$

we obtain for the bracket symbols $\begin{Bmatrix}\mu\nu\\ s\end{Bmatrix}$ the following values, whereby those that vanish are omitted:

$$\begin{Bmatrix}11\\ 1\end{Bmatrix} = \frac{1}{2}\frac{M'}{M}, \qquad \begin{Bmatrix}22\\ 1\end{Bmatrix} = -\frac{r}{M}, \qquad \begin{Bmatrix}33\\ 1\end{Bmatrix} = -\frac{r}{M}\sin^2\vartheta,$$

$$\begin{Bmatrix}44\\ 1\end{Bmatrix} = -\frac{1}{2}\frac{W'}{M}, \qquad \begin{Bmatrix}12\\ 2\end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix}33\\ 2\end{Bmatrix} = -\sin\vartheta\cos\vartheta,$$

$$\begin{Bmatrix}13\\ 3\end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix}23\\ 3\end{Bmatrix} = \cot\vartheta, \qquad \begin{Bmatrix}14\\ 4\end{Bmatrix} = \frac{1}{2}\frac{W'}{W}.$$

With these we form:

$$\begin{aligned}
K_{11} = {} & \frac{\partial}{\partial r}\left(\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}13\\3\end{Bmatrix}+\begin{Bmatrix}14\\4\end{Bmatrix}\right)-\frac{\partial}{\partial r}\begin{Bmatrix}11\\1\end{Bmatrix}\\
& +\begin{Bmatrix}11\\1\end{Bmatrix}\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}\begin{Bmatrix}21\\2\end{Bmatrix}+\begin{Bmatrix}13\\3\end{Bmatrix}\begin{Bmatrix}31\\3\end{Bmatrix}+\begin{Bmatrix}14\\4\end{Bmatrix}\begin{Bmatrix}41\\4\end{Bmatrix}\\
& -\begin{Bmatrix}11\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}13\\3\end{Bmatrix}+\begin{Bmatrix}14\\4\end{Bmatrix}\right)\\
= {} & \frac{1}{2}\frac{W''}{W}+\frac{1}{4}\frac{W'^2}{W^2}-\frac{M'}{rM}-\frac{1}{4}\frac{M'W'}{MW}
\end{aligned}$$

$$\begin{aligned}
K_{22} = {} & \frac{\partial}{\partial\vartheta}\begin{Bmatrix}23\\3\end{Bmatrix}-\frac{\partial}{\partial r}\begin{Bmatrix}22\\1\end{Bmatrix}\\
& +\begin{Bmatrix}21\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix}+\begin{Bmatrix}22\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}23\\3\end{Bmatrix}\begin{Bmatrix}32\\3\end{Bmatrix}\\
& -\begin{Bmatrix}22\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}13\\3\end{Bmatrix}+\begin{Bmatrix}14\\4\end{Bmatrix}\right)\\
= {} & -1+\frac{1}{2}\frac{rM'}{M^2}+\frac{1}{M}+\frac{1}{2}\frac{rW'}{MW}
\end{aligned}$$ [69]

$$\begin{aligned}
K_{33} = {} & -\frac{\partial}{\partial r}\begin{Bmatrix}33\\1\end{Bmatrix}-\frac{\partial}{\partial\vartheta}\begin{Bmatrix}33\\2\end{Bmatrix}\\
& +\begin{Bmatrix}31\\3\end{Bmatrix}\begin{Bmatrix}33\\1\end{Bmatrix}+\begin{Bmatrix}32\\3\end{Bmatrix}\begin{Bmatrix}33\\2\end{Bmatrix}+\begin{Bmatrix}33\\1\end{Bmatrix}\begin{Bmatrix}13\\3\end{Bmatrix}+\begin{Bmatrix}33\\2\end{Bmatrix}\begin{Bmatrix}23\\3\end{Bmatrix}\\
& -\begin{Bmatrix}33\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}13\\3\end{Bmatrix}+\begin{Bmatrix}14\\4\end{Bmatrix}\right)-\begin{Bmatrix}33\\2\end{Bmatrix}\begin{Bmatrix}23\\3\end{Bmatrix}\\
= {} & \sin^2\vartheta\left(-1-\frac{1}{2}\frac{rM'}{M^2}+\frac{1}{M}+\frac{1}{2}\frac{rW'}{MW}\right)
\end{aligned}$$

$$
\begin{aligned}
K_{44} &= -\frac{\partial}{\partial r}\begin{Bmatrix}44\\1\end{Bmatrix} + \begin{Bmatrix}41\\4\end{Bmatrix}\begin{Bmatrix}44\\1\end{Bmatrix} + \begin{Bmatrix}44\\1\end{Bmatrix}\begin{Bmatrix}41\\4\end{Bmatrix} \\
&\quad - \begin{Bmatrix}44\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}13\\3\end{Bmatrix} + \begin{Bmatrix}14\\4\end{Bmatrix}\right) \\
&= \frac{1}{2}\frac{W''}{M} - \frac{1}{4}\frac{M'W'}{M^2} - \frac{1}{4}\frac{W'^2}{MW} + \frac{W'}{rM}
\end{aligned}
$$

$$
\begin{aligned}
K &= \sum_s g^{ss}K_{ss} = \frac{W''}{MW} - \frac{1}{2}\frac{W'^2}{MW^2} - 2\frac{M'}{rM^2} - \frac{1}{2}\frac{M'W'}{M^2W} \\
&\qquad - \frac{2}{r^2} + \frac{2}{r^2M} + 2\frac{W'}{rMW}.
\end{aligned}
$$

Because

$$
\sqrt{g} = \sqrt{MW}\,r^2\sin\vartheta
$$

we have

$$
K\sqrt{g} = \left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)' - 2\frac{rM'\sqrt{W}}{M^{3/2}} - 2\sqrt{MW} + 2\sqrt{\frac{W}{M}}\right\}\sin\vartheta,
$$

and if we put

$$
M = \frac{r}{r-m}, \qquad W = w^2\frac{r-m}{r},
$$

where now m and w are the unknown functions of r, we finally obtain

$$
K\sqrt{g} = \left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)' - 2wm'\right\}\sin\vartheta,
$$

[70] | so that the variation of the quadruple integral

$$
\iiiint K\sqrt{g}\;dr\;d\vartheta\;d\varphi\;dl
$$

is equivalent to the variation of the single integral

$$
\int wm'dr
$$

and leads to the Lagrange equations

$$
\begin{aligned}
m' &= 0 \\
w' &= 0.
\end{aligned}
\qquad (44)
$$

It is easy to convince oneself that these equations indeed imply that all $K_{\mu\nu}$ vanish; they therefore represent essentially the most general solution of equations (36) under the assumptions 1., 2., 3., we made. If we take as integrals of (44) $m = \alpha$, where α is a constant, and $w = 1$, which evidently is no essential restriction, then for $l = it$ (43) results in the desired metric in the form first found by Schwarzschild

$$G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha}dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \frac{r-\alpha}{r}dl^2. \qquad (45)$$

The *singularity* of the metric at $r = 0$ disappears only if we take $\alpha = 0$, i.e. *the metric of the pseudo-Euclidean geometry is the only regular metric that corresponds to a world without electricity under the assumptions* 1., 2., 3.

If $\alpha \neq 0$, then $r = 0$ and, for positive α also $r = \alpha$, prove to be places where the metric is not regular. Here I call a metric or gravitational field $g_{\mu\nu}$ *regular* at some place if it is possible to introduce by transformation with unique inverse a coordinate system for which the corresponding functions $g'_{\mu\nu}$ at that place are regular, that is they are continuous and arbitrarily differentiable at the place and its neighborhood, and have a determinant g' that differs from zero.

Although in my view only regular solutions of the basic physical equations represent reality directly, still it is precisely the solutions with places of non-regularity that are an important mathematical instrument for approximating characteristic regular solutions—and in this sense, following Einstein and Schwarzschild, the metric (45), not regular at $r = 0$ and $r = \alpha$, is to be viewed as the expression for | gravity of a [71]
centrally symmetric mass distribution in the neighborhood of the origin[7]. In the same sense a point mass is to be understood as the limit of a certain distribution of electricity about one point, but I refrain at this place from deriving its equations of motion from my basic physical equations. A similar situation prevails for the question about the differential equations for the propagation of light.

Following Einstein, let the *following two axioms* serve as a substitute for a derivation from the basic equations:

The motion of a point mass in a gravitational field is described by a geodesic line, which is a time line[8].

The motion of light in a gravitational field is described by a geodesic null line.

Because the world line representing the motion of a point mass shall be a time line, it is easily seen to be always possible to bring the point mass to rest by *true* spacetime transformations, i.e. there are *true* spacetime coordinate systems with respect to which the point mass remains at rest.

The differential equations of geodesic lines for the centrally symmetric gravitational field (45) arise from the variational problem

---

7 To transform the locations $r = \alpha$ to the origin, as Schwarzschild does, is not to be recommended in my opinion; Schwarzschild's transformation is moreover not the simplest that achieves this goal.

8 This last restrictive addition is to be found neither in Einstein nor in Schwarzschild.

$$\delta\int\left(\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2+r^2\left(\frac{d\vartheta}{dp}\right)^2+r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2-\frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2\right)dp = 0,$$

and become, by well-known methods:

$$\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2+r^2\left(\frac{d\vartheta}{dp}\right)^2+r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2-\frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = A, \tag{46}$$

$$\frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right)-r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dp}\right)^2 = 0, \tag{47}$$

$$r^2\sin^2\vartheta\frac{d\varphi}{dp} = B, \tag{48}$$

$$\frac{r-\alpha}{r}\frac{dt}{dp} = C, \tag{49}$$

where A, B, C denote constants of integration. |

[72] I first prove *that the orbits in the* $r\vartheta\varphi$- *space always lie in planes passing through the center of the gravitation*.

To this end we eliminate the parameter p from the differential equations (47) and (48) to obtain a differential equation for ϑ as a function of φ. We have the identity

$$\begin{aligned}\frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right) &= \frac{d}{dp}\left(r^2\frac{d\vartheta}{d\varphi}\cdot\frac{d\varphi}{dp}\right)\\ &= \left(2r\frac{dr}{d\varphi}\frac{d\vartheta}{d\varphi}+r^2\frac{d^2\vartheta}{d\varphi^2}\right)\left(\frac{d\varphi}{dp}\right)^2+r^2\frac{d\vartheta}{d\varphi}\frac{d^2\varphi}{dp^2}.\end{aligned} \tag{50}$$

On the other hand, differentiation of (48) with respect to p gives:

$$\left(2r\frac{dr}{d\varphi}\sin^2\vartheta+2r^2\sin\vartheta\cos\vartheta\frac{d\vartheta}{d\varphi}\right)\left(\frac{d\varphi}{dp}\right)^2+r^2\sin^2\vartheta\frac{d^2\varphi}{dp^2} = 0,$$

and if we take from this the value of $\dfrac{d^2\varphi}{dp^2}$ and substitute on the right of (50), it becomes

$$\frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right) = \left(\frac{d^2\vartheta}{d\varphi^2}-2\cot\vartheta\left(\frac{d\vartheta}{d\varphi}\right)^2\right)r^2\left(\frac{d\varphi}{dp}\right)^2.$$

Thus equation (47) takes the form:

$$\frac{d^2\vartheta}{d\varphi^2}-2\cot\vartheta\left(\frac{d\vartheta}{d\varphi}\right)^2 = \sin\vartheta\cos\vartheta,$$

a differential equation whose general integral is

$$\sin\vartheta\cos(\varphi + a) + b\cos\vartheta = 0,$$

where a and b denote constants of integration.

This provides the desired proof, and it is therefore sufficient for further discussion of geodesic lines to consider only the value $\vartheta = 2/\pi$. Then the variational problem simplifies as follows

$$\delta\int\left\{\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2\right\}dp = 0,$$

and the three differential equations of first order that arise from it are |

$$\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = A, \qquad (51)$$ [73]

$$r^2\frac{d\varphi}{dp} = B, \qquad (52)$$

$$\frac{r-\alpha}{r}\frac{dt}{dp} = C. \qquad (53)$$

The Lagrange differential equation for r

$$\frac{d}{dp}\left(\frac{2r}{r-\alpha}\frac{dr}{dp}\right) + \frac{\alpha}{(r-\alpha)^2}\left(\frac{dr}{dp}\right)^2 - 2r\left(\frac{d\varphi}{dp}\right)^2 + \frac{\alpha}{r^2}\left(\frac{dt}{dp}\right)^2 = 0 \qquad (54)$$

is necessarily related to the above equations, in fact if we denote the left sides of (51), (52), (53), (54) with [1], [2], [3], [4] respectively we have identically

$$\frac{d[1]}{dp} - 2\frac{d\varphi}{dp}\frac{d[2]}{dp} + 2\frac{dt}{dp}\frac{d[3]}{dp} = \frac{dr}{dp}[4]. \qquad (55)$$

By choosing $C = 1$, which amounts to multiplying the parameter p by a constant, and then eliminating p and t from (51), (52), (53) we obtain that differential equation for $\rho = 1/r$ as a function of φ found by Einstein and Schwarzschild, namely:

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1+A}{B^2} - \frac{A\alpha}{B^2}\rho - \rho^2 + \alpha\rho^3. \qquad (56)$$

This equation represents the orbit of the point mass in polar coordinates; in first approximation for $\alpha = 0$ with $B = \sqrt{\alpha}b$, $A = -1 + \alpha a$ the Kepler motion follows from it, and the second approximation than leads to the most shining discovery of the present: the calculation of the advance of the perihelion of Mercury.

According to the axiom above the world line for the motion of a point mass shall be a time line; from the definition of the time line it thus follows that always $A < 0$.

We now ask in particular whether a circle, i.e. $r = \text{const.}$ can be the orbit of a motion. The identity (55) shows that in this case—because of $dr/dp = 0$—equation (54) is by no means a consequence of (51), (52), (53); the latter three equations therefore are insufficient to determine the motion; instead the necessary equations to be satisfied are (52), (53), (54). From (54) it follows that |

[74]
$$-2r\left(\frac{d\varphi}{dp}\right)^2 + \frac{\alpha}{r^2}\left(\frac{dt}{dp}\right)^2 = 0 \tag{57}$$

or that for the speed v on the circular orbit

$$v^2 = \left(r\frac{d\varphi}{dt}\right)^2 = \frac{\alpha}{2r}. \tag{58}$$

On the other hand, since $A < 0$, (51) implies the inequality

$$r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 < 0 \tag{59}$$

or by using (57)

$$r > \frac{3\alpha}{2}. \tag{60}$$

With (58) this implies the inequality for the speed of the mass point moving on a circle[9]

$$v < \frac{1}{\sqrt{3}}. \tag{61}$$

The inequality (60) allows the following interpretation: From (58) the angular speed of the orbiting point mass is

$$\frac{d\varphi}{dt} = \sqrt{\frac{\alpha}{2r^3}}.$$

So if we want to introduce instead of r, φ the polar coordinates of a coordinate system co-rotating about the origin, we only have to replace

$$\varphi \quad \text{by} \quad \varphi + \sqrt{\frac{\alpha}{2r^3}}\,t.$$

After the corresponding spacetime transformation the metric

9 Schwarzschild's (loc. cit.) claim that the speed of the point mass on a circular orbit approaches the limit $1/\sqrt{2}$ as the orbit radius is decreased corresponds to the inequality $r \geq \alpha$ and should not be regarded as accurate, according to the above.

$$\frac{r}{r-\alpha}dr^2 + r^2 d\varphi^2 - \frac{r-\alpha}{r}dt^2$$

becomes

$$\frac{r}{r-\alpha}dr^2 + r^2 d\varphi^2 + \sqrt{2\alpha r}\, d\varphi\; dt + \left(\frac{\alpha}{2r} - \frac{r-\alpha}{r}\right)dt^2.$$

| Here the inequality $g_{44} < 0$ is satisfied due to (60), and since the other inequalities (31) are satisfied, *the transformation under discussion of the point mass to rest is a true spacetime transformation*. [75]

On the other hand, the upper limit $1/\sqrt{3}$ found in (61) for the speed of a mass point on a circular orbit also has a simple interpretation. According to the axiom for light propagation this propagation is represented by a null geodesic. Accordingly if we put $A = 0$ in (51), instead of the inequality (59) the result for circular light propagation is the equation

$$r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = 0;$$

together with (57) this implies for the radius of the light's orbit:

$$r = \frac{3\alpha}{2}$$

and for the speed of the orbiting light the value that occurs as the upper limit in (61):

$$v = \frac{1}{\sqrt{3}}.$$

In general we find for the orbit of light from (56) with $A = 0$ the differential equation

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1}{B^2} - \rho^2 + \alpha\rho^3; \tag{62}$$

for $B = \frac{3\sqrt{3}}{2}\alpha$ it has the circle $r = \frac{3\alpha}{2}$ as a Poincaré "cycle"—corresponding to the circumstance that thereupon $\rho - \frac{2}{3\alpha}$ is a double factor of the right-hand side. Indeed in this case—and correspondingly for the more general equation (56)—the differential equation (62) possesses infinitely many integral curves, which approach that circle as the limit of spirals, as demanded by Poincaré's general theory of cycles.

If we consider a light ray approaching from infinity and take α small compared to the ray's distance of closest approach from the center of gravitation, then the light ray has approximately the form of a hyperbola with focus at the center.[10] |

[76] A counterpart to the motion on a circle is the motion on a straight line that passes through the center of gravitation. We obtain the differential equation for this motion if we set $\varphi = 0$ in (54) and then eliminate p from (53) and (54); the differential equation so obtained for r as a function of t is

$$\frac{d^2r}{dt^2} - \frac{3\alpha}{2r(r-\alpha)}\left(\frac{dr}{dt}\right)^2 + \frac{r(r-\alpha)}{2r^3} = 0 \tag{63}$$

with the integral following from (51)

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r-\alpha}{r}\right)^2 + A\left(\frac{r-\alpha}{r}\right)^3. \tag{64}$$

According to (63) the acceleration is negative or positive, i.e. gravitation acts attractive or repulsive, according as the absolute value of the velocity

$$\left|\frac{dr}{dt}\right| < \frac{1}{\sqrt{3}}\frac{r-\alpha}{r}$$

or

$$> \frac{1}{\sqrt{3}}\frac{r-\alpha}{r}.$$

For light we have because of (64)

$$\left|\frac{dr}{dt}\right| = \frac{r-\alpha}{r};$$

light propagating in a straight line towards the center is always repelled, in agreement with the last inequality; its speed increases from 0 at $r = \alpha$ to 1 at $r = \infty$.

When α as well as dr/dt are small, (63) becomes approximately the Newtonian equation

$$\frac{d^2r}{dt^2} = -\frac{\alpha}{2}\frac{1}{r^2}.$$

10 A detailed discussion of the differential equations (56) and (62) will be the task of a communication by V. Fréedericksz to appear in these pages.